Thomae formulas in application to finding reality conditions for integrable hierarchies

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Spectral curves of the KdV, sine-Gordon, and mKdV hierarchies all belong to the family of hyperelliptic curves of the form

$$\mathcal{C}: \qquad f(x,y) \equiv -y^2 + \mathcal{P}(x) \equiv -y^2 + x^{2g+1} + \textstyle\sum_{i=1}^{2g} \lambda_{2i+2} x^{2g-i} = 0, \tag{1}$$

where the genus g of $\mathcal{C}$ coincides with the number of gaps of a hamiltonian systems in the hierarchy. Parameters $\lambda = \{\lambda_{2i+2}\}_{i=1}^{2g}$ serve as integrals of motion.

Let $\mathrm{Jac}(\mathcal{C}) = \mathbb{C}^g/\{\omega, \omega'\}$ be the Jacobian variety of $\mathcal{C}$ with respect to the lattice generated by columns of not normalized period matrices $\omega = (\omega_{i,j})$, $\omega' = (\omega'_{i,j})$ defined by

$$\omega_{i,j} = \int_{\mathfrak{a}_j} \mathrm{d}u_{2i-1}, \qquad \omega'_{i,j} = \int_{\mathfrak{b}_j} \mathrm{d}u_{2i-1}, \quad \text{where} \quad \mathrm{d}u_{2i-1} = \frac{x^{g-i}\mathrm{d}x}{-2y}.$$

Second kind period matrices $\eta = (\eta_{i,j})$, $\eta' = (\eta'_{i,j})$ are obtained from the second kind differentials associated with the first kind differentials $\mathrm{d}u = (\mathrm{d}u_1, \mathrm{d}u_3, \ldots, \mathrm{d}u_{2g-1})^t$ defined above.

Each curve $\mathcal{C}$ is uniformized by means of the multiply periodic $\wp$-functions

$$\wp_{i,j}(u) = -\frac{\partial^2 \log \sigma(u)}{\partial u_i \partial u_j}, \qquad \wp_{i,j,k}(u) = -\frac{\partial^3 \log \sigma(u)}{\partial u_i \partial u_j \partial u_k},$$

which generalize the Weierstrass $\wp$-function to higher genera. The sigma function is defined by

$$\sigma(u) = C \exp\big(-\tfrac{1}{2} u^t \varkappa u\big) \theta[K](\omega^{-1}u; \omega^{-1}\omega'), \tag{2}$$

see [1, Eq.(2.3)], where $[K]$ is the characteristic of the vector K of Riemann constants, and $\varkappa = \eta\omega^{-1}$.

The mentioned completely integrable equations have the following finite-gap solutions, $b \in \mathbb{R}$, $c_i \in \mathbb{R}$,

KdV	$w_t = 6ww_x + w_{xxx}$	$w(x,t) = -b\wp_{1,1}(u), \quad u = -b(x, t, c_5, \ldots, c_{2g-1})^t + \omega K,$
sine-Gordon	$\phi_{t,x} = 4\sin\phi$	$\phi(x,t) = \imath \log\big(-\lambda_{4g}^{-1/2} \wp_{1,2g-1}(u)\big), \; u = \imath b(x, c_3, \ldots, c_{2g-3}, t)^t + \omega K,$
sinh-Gordon	$\phi_{t,x} = -4\sinh\phi$	$\phi(x,t) = \log\big(-\lambda_{4g}^{-1/2} \wp_{1,2g-1}(u)\big), \; u = b(x, c_3, \ldots, c_{2g-3}, t)^t + \omega K,$
mKdV	$w_t = 6\varsigma w^2 w_x - w_{xxx}$	$w(x,t) = -\dfrac{\mathrm{b}\wp_{1,1,2N-1}(u)}{2\wp_{1,2N-1}(u)}, \quad u = \mathrm{b}(x, -4\mathrm{b}^2 t, c_5, \ldots, c_{2g-1})^t + \omega K.$

In the case of defocusing mKdV, assign $\varsigma = 1$, $\mathrm{b} = b$. In the case of focusing mKdV, $\varsigma = -1$, $\mathrm{b} = \imath b$.

The reality conditions require all solutions to be real-valued and bounded functions of real variables x, and t. That is, the reality conditions are specified by the choice of a path in $\mathrm{Jac}(\mathcal{C})$ where a solution of the system in question is real-valued. An answer to this question is obtained from the analysis of values of the σ-function at half-periods on the spectral curve $\mathcal{C}$.

Recall, that after separation of variables, a g-gap hamiltonian system in one of the mentioned hierarchies splits into g one-particle systems, whose phase trajectories are determined by (1), namely by $f(x_i, y_i) = 0$, $i = 1, \ldots, g$, where x_i serves as the coordinate, and y_i as the momentum. Thus, $-\mathcal{P}(x)$ serves as the potential, and so roots of $\mathcal{P}$ serve as turn points. Therefore, the Abel image of a divisor composed from g points, one on each one-particle trajectory, goes through half-periods. For the purpose of a bounded solution, these half-periods should be non-singular.

The Thomae formulas introduce a connection between null values of the theta function with characteristics (or its first non-vanishing derivative), called theta constants (or theta derivatives), on the one hand, and x-coordinates of the branch points which produce half-periods corresponding to the characteristics, on the other hand. Instead of theta constants and theta derivatives we use values of the σ-function (or its first non-vanishing derivative) at half-periods. Each half-period is represented by a partition on the set of indices of branch points.

Let $\mathcal{S} = \{0, 1, 2, \ldots, 2g+1\}$ be the set of indices of all branch points, and 0 stands for infinity. A partition $\mathcal{I}_0 \cup \mathcal{J}_0 = \mathcal{S}$, $\mathrm{card}\mathcal{I}_0 = \mathrm{card}\mathcal{J}_0 = g+1$, represents a characteristic of multiplicity 0, or an even non-singular characteristic, which describes a half-period Ω_0 such that $\sigma(\Omega_0) \neq 0$. A partition $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m} = \mathcal{S}$, $\mathrm{card}\mathcal{I}_\mathfrak{m} = g+1-2\mathfrak{m}$, $\mathrm{card}\mathcal{J}_\mathfrak{m} = g+1+2\mathfrak{m}$, represents a characteristic of multiplicity $\mathfrak{m}$, which describes a half-period $\Omega_\mathfrak{m}$ such that $\partial_{u_1}^{\mathfrak{m}}\sigma(\Omega_\mathfrak{m}) \neq 0$ and $\partial_{u_1}^{\mathfrak{r}}\sigma(\Omega_\mathfrak{m}) = 0$ if $0 \leqslant \mathfrak{r} < \mathfrak{m}$. All half-periods represented by partitions $\mathcal{I}_\mathfrak{m} \cup \mathcal{J}_\mathfrak{m} = \mathcal{S}$ with $\mathfrak{m} > 0$ are called singular, due to $\wp$-functions have singularities at such half-periods.

In the fundamental domain of $\mathrm{Jac}(\mathcal{C})$, there exist 2^{2g} half-periods. In the case of a real curve (with real parameters λ), these half-periods form 2^g lines parallel to the real axes, and 2^g lines parallel to the imaginary axes, each line contains 2^g half-periods. It is proven, see [2, Propositions 2, 3], [3, Theorem 4], that there exists only one line parallel to the real axes, and only one line parallel to the imaginary axes, which contains no singular half-periods. Any of the two lines can serve as the domain for finite-gap solutions of the integrable systems.

Further, the reality conditions require real values of solutions, that is for $s \in \mathbb{R}^g$

KdV	$\wp_{1,1}(s + \omega K) \in \mathbb{R}$,
sine-Gordon	$\lvert\wp_{1,2g-1}(\imath s + \omega K)\rvert^2 = \lambda_{4g}$,
sinh-Gordon	$\lvert\wp_{1,2g-1}(s + \omega K)\rvert^2 = \lambda_{4g}$,
defocusing mKdV	$\dfrac{\wp_{1,1,2N-1}(s + \omega K)}{\wp_{1,2N-1}(s + \omega K)} \in \mathbb{R}$,
focusing mKdV	$\imath\dfrac{\wp_{1,1,2N-1}(\imath s + \omega K)}{\wp_{1,2N-1}(\imath s + \omega K)} \in \mathbb{R}$.

Direct computations of the above expressions show that not all real curves can serve as spectral curves of the mentioned integrable hierarchies. As proven in [2, Propositions 4, 5], [3, Theorem 5], [4, Theorem 5], the requires reality conditions are satisfied on the following curves.

Theorem 1. *Hyperelliptic curves which possess a branch point at infinity, and all other branch points are real, serve as spectral curves for the KdV hierarchy.*

Theorem 2. *There exist two types of real hyperelliptic curves which satisfy the reality conditions for the sine(sinh)-Gordon equation and the mKdV equation:*

- *(RC) hyperelliptic curves which possess a branch point at infinity, a branch point at the origin, and all other branch points are real.*
- *(IC) hyperelliptic curves which possess a branch point at infinity, a branch point at the origin, and all other branch points split in complex conjugate pairs.*

Curves (RC) serve as spectral for the sinh-Gordon, and defocusing mKdV hierarchies. Curves (IC) serve as spectral for the sine-Gordon, and focusing mKdV hierarchies.

References

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