On the derivations and automorphisms of Clifford algebras over countable-dimensional vector spaces

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Let $\mathcal{C}\ell(V,f)$ denote the Clifford algebra of a vector space V over a field $\mathbb{F}$ of characteristic not equal to 2, generated by V with unit 1 and defining relations $v^2 = f(v)\cdot 1$, where f is a nondegenerate quadratic form; see [4, 5].

Assume the ground field $\mathbb{F}$ is algebraically closed. According to N. Jacobson [2], if the dimension of the vector space V is even, then the Clifford algebra $\mathcal{C}\ell(V,f)$ is isomorphic to a matrix algebra; if the dimension of V is odd, then $\mathcal{C}\ell(V,f)$ is isomorphic to the direct sum of matrix algebras. For an infinite dimensional vector space V, the Clifford algebra $Cl(V,f)$ is a locally matrix algebra; see [1].

Two main families of derivations and automorphisms of Clifford algebras are known:

(1) Inner derivations and inner automorphisms.
(2) Bogolyubov derivations and Bogolyubov automorphisms.

The Clifford algebra $\mathcal{C}\ell(V,f)$ is graded by the cyclic group of order 2, expressed as $\mathcal{C}\ell(V,f) = \mathcal{C}\ell(V,f)_{\bar{0}} + \mathcal{C}\ell(V,f)_{\bar{1}}$. A derivation D of the algebra $\mathcal{C}\ell(V,f)$ is called *even* if $D(\mathcal{C}\ell(V,f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{0}}$, $D(\mathcal{C}\ell(V,f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{1}}$; and *odd* if $D(\mathcal{C}\ell(V,f)_{\bar{0}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{1}}$, $D(\mathcal{C}\ell(V,f)_{\bar{1}}) \subseteq \mathcal{C}\ell(V,f)_{\bar{0}}$.

We describe derivations of the Clifford algebra associated with a nondegenerate quadratic form on a countable-dimensional vector space over an algebraically closed field of characteristic not equal to 2. Any nonzero derivation D of $\mathcal{C}\ell(V,f)$ can be uniquely represented as a sum: $D = \sum_S \alpha_S \,\mathrm{ad}(v_S)$, where $0 \neq \alpha_S \in \mathbb{F}$.

- For an *even* derivation D, the subsets S are finite, nonempty subsets of $\mathbb{N}$ of even order, and each $i \in \mathbb{N}$ belongs to at most finitely many subsets S.
- For an *odd* derivation D, the subsets S are finite subsets of $\mathbb{N}$ of odd order, and each $i \in \mathbb{N}$ lies in all but finitely many subsets S.

Additionally, we characterize when a nonzero even derivation of the Clifford algebra is a Bogolyubov derivation and when a Bogolyubov derivation corresponding to a skew-symmetric linear transformation is an inner derivation.

Now, suppose the field $\mathbb{F} = \mathbb{R}$ is the field of real numbers, and let $f : V \to \mathbb{R}$ be a positive definite quadratic form. In this case, the Clifford algebra $\mathcal{C}\ell(V,f)$ naturally inherits the structure of a normed algebra. In a 2022 MathOverflow discussion, M. Ludewig (see [3]) posed the question of whether every automorphism of $\mathcal{C}\ell(V,f)$ is continuous with respect to this norm.

In response, we construct an algebraic automorphism of $\mathcal{C}\ell(V,f)$ that is not continuous with respect to the given norm.

References

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