From Maxwell's equations to relativistic Schrödinger equation via Schwartz Linear Algebra and Killing vector fields on the 2-sphere

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In this work, we develop a comprehensive mathematical framework unifying scalar relativistic quantum mechanics with classical electromagnetic field theory by means of Schwartz-linear algebra. Building upon the foundations introduced by David Carfì in [1, 2, 3, 4], we construct a partial embedding of tempered scalar distributions into spaces of tempered vector-valued fields that carry natural Maxwellian structure.

The key object of our study is an embedding operator $J_{(\eta,f)}$, Schwartz-linear and continuous, that maps a large class of complex wave distributions

$$\psi \in \mathcal{S}'(\mathbb{M}^4, \mathbb{C})$$

into transverse vector fields

$$F \in W = \mathcal{S}'(\mathbb{M}^4, \mathbb{C}^3)$$

via spectral synthesis with polarization. Specifically, the embedding is constructed using a transverse, right-handed polarization frame

$$f : k \mapsto f(k) = (r(k), s(k)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

defined on the dual space $\mathbb{M}_4^*$ minus Π, where Π is a singular plane and r is a killing vector field on the 2-sphere extended omogeneously to the whole dual of Minkovski space-time minus Π. The Maxwell's basis

$$w : k \mapsto w_k := \eta_k(r(k) + is(k)),$$

with $\eta_k(x) = e^{i\langle k,x\rangle}$, forms a Schwartz linearly independent system of circularly polarized plane waves, generating a vast subspace S of Schwartz-Maxwell electromagnetic field space W. The map

$$J_{(\eta,f)} : \psi \mapsto J_{(\eta,f)}(\psi) = \int_{\mathbb{M}_4^*} (\psi)_\eta w$$

embeds scalar wave distributions into the Maxwell-Schwartz field space, provided that the complex wave distribution ψ admits a momentum representation $(\psi)_\eta$ vanishing around the singular plane Π.

We show that this embedding preserves eigenstructures of quantum observables diagonale on η. The momentum operator $\hat{p} = -i\hbar\nabla$ and energy operator $\hat{E} = i\hbar\partial_0$ act compatibly through $J_{(\eta,f)}$, and w_k are simultaneous eigenfunctions of $\hat{p}$ and curl, with eigenvalues $\hbar\vec{k}$ and $|\vec{k}|$, respectively. The operator $\hbar\,\mathrm{curl}$ is therefore identified with the momentum magnitude operator on the subspace S of the Maxwell-Schwartz space.

Furthermore, the theory is extended to general embeddings $J_{(\beta,f)}$, constructed from arbitrary Schwartz bases β and smooth frame fields $f : D \to \mathbb{C}^3$ with the same domain of β. These embeddings commute with all observables diagonal in the basis β, yielding a functorial structure. We also define position-space embeddings $I_j (j = 1, 2, 3)$ using Dirac delta basis and demonstrate that they preserve position eigenstates. This duality between frequency-space and position-space embeddings reveals a deep symmetry between quantum representations.

As an example, a geometric interpretation is introduced via the use of Frenet frames along spatial curves, allowing for the representation of localized electromagnetic waves carrying geometric signatures of trajectories. Fields such as

$$\delta_0 \circ \gamma \cdot f \circ \gamma : t \mapsto \delta_{\gamma(t)} \cdot f(\gamma(t))$$

are shown to encode the curve γ via the support and polarization f.

The relativistic Schrödinger equation for photons, in tempered distribution space, is recovered in the form

$$\hat{E}\psi = c|\hat{p}|\psi.$$

We show that in our subspace S the curl Maxwell's equations can be synthesized into the same Schrödinger's form equation

$$\hat{E}F = c|\hat{p}|F,$$

where $\hat{E}$ is the energy operator in our space W, perfectly analogous to the energy operator in the space of complex tempered distribution, $\hat{p}$ is the momentum operator in W whose magnitude operator equals the operator $\hbar$ curl on the subspace S. This shows that any wave distribution ψ, with a momentum representation vanishing around the singular plane Π, can be smoothly interpreted as encoding an electromagnetic-type field. A wave distribution ψ solves the massless relativistic Schrödinger equation if an only if the corresponding electromagnetic-type field solves the massless Schrödinger-Maxwell equation in W. Analogously, we construct a faithfull representation of the relativistic Schrödinger equation for massive particles in our space W, showing that each wave distribution state of a massive particle (complex field) can be smoothly interpreted as an electromagnetic-like field in W.

Delta distributions

$$\psi(x,t) = \delta(x \mp ct)$$

are proven to be solutions of photons equation with spectral support positive or negative, corresponding to right-moving and left-moving massless particles, respectively. The relation

$$m = \hbar|\vec{k}|/c$$

defines the relativistic mass of a photon as a function of spectral content. On the other hand, the dispersion relation of the massive plane wave fields satisfying the Maxwell-Schrödinger equation, is given by the Einstein's energy relation.

This work lays a foundation for a full spectral theory of relativistic fields within tempered distribution spaces, connecting canonical Quantum Mechanics, Maxwell's equations, and geometric field structures under a unified, mathematically rigorous umbrella.

References

[1] David Carfì. *Quantum Mechanics and Dirac Calculus in Schwartz Distribution Spaces*, volume 1,2,3. Il Gabbiano, 2022.

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