Infinitesimal conformal transformations and vielbein formalism

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When exploring infinitesimal transformations of differentiable manifolds, we typically use holonomic coordinate systems. However, to study spinor fields, we introduce a set of four independent vector fields $t^i_a(x)$, with $a = 0, 1, 2, 3$, defined at each point of a spacetime $(V^{1,3}, g)$. These vectors are orthonormal with respect to the spacetime metric and satisfy the condition:

$$t^i_a(x)\, t^j_b(x)\, g_{ij}(x) = \eta_{ab}, \quad \text{where} \quad \eta_{ab} = \mathrm{diag}(1, -1, -1, -1).$$

The inverse matrix $t^a_i(x)$ is defined such that:

$$t^i_a(x)\, t^a_j(x) = \delta^i_j, \quad t^i_a(x)\, t^b_i(x) = \delta^a_b.$$

This approach is known as the *vielbein formalism*, where the field $t^a_i(x)$ is called the *vielbein* [1].

The spin connection is defined by the following expression:

$$\omega_k{}^a{}_b = \left(t^i_b\, \Gamma^h_{ki} + \partial_k t^h_b \right) t^a_h. \tag{1}$$

From equation (1), we obtain the identity:

$$\partial_k t^h_a + \Gamma^h_{jk} t^j_a - \omega_k{}^b{}_a t^h_b = 0.$$

The covariant derivative of a spinor field $\psi(x)$ is given by:

$$\nabla_k \psi = \partial_k \psi - \frac{1}{4} \omega_{kab}\, \gamma^{ab} \psi = \partial_k \psi + \Gamma_k \psi,$$

where $\gamma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a)$ is the antisymmetrized product of two gamma matrices.

The covariant derivative of the adjoint spinor $\overline{\psi} = \psi^\dagger \gamma^0$ is:

$$\nabla_k \overline{\psi} = \partial_k \overline{\psi} + \overline{\psi} \frac{1}{4} \omega_{kab} \gamma^{ab} = \partial_k \overline{\psi} - \overline{\psi} \Gamma_k.$$

Infinitesimal transformations of the form

$$\overline{x}^h = x^h + \epsilon \xi^h(x^1, x^2, \ldots, x^n)$$

are called *conformal transformations* if the following condition is satisfied [4, p. 157]:

$$\mathfrak{L}_\xi g_{ij} = \xi_{i,j} + \xi_{j,i} = \varphi\, g_{ij}, \tag{2}$$

where $\varphi(x)$ is a scalar function.

Taking the Lie derivative of the vielbein yields:

$$\mathfrak{L}_\xi t^a_i(x) = \frac{\varphi}{2} t^a_i(x). \tag{3}$$

For any geometric object field $\Omega^{\Lambda}_{M}(\xi)$, the following identity holds [4, p. 23]:

$$\mathfrak{L}_{\xi}\partial_k\Omega^{\Lambda}_{M}(\xi) = \partial_k\mathfrak{L}_{\xi}\Omega^{\Lambda}_{M}(\xi). \tag{4}$$

Using this result, we find the Lie derivative of the spin connection:

$$\mathfrak{L}_{\xi}\omega_{kab} = \frac{1}{2}\left(t_{ka}\varphi_b - t_{kb}\varphi_a\right),$$

where $\varphi_b = \partial_b\varphi = t^j_b\partial_j\varphi$.

Thus, for the spin-affine connection Γ_k, we obtain:

$$\mathfrak{L}_{\xi}\Gamma_k = -\frac{1}{8}(t_{ka}\varphi_b - t_{kb}\varphi_a)\gamma^{ab} = -\frac{1}{4}t_{ka}\varphi_b\gamma^{ab}. \tag{5}$$

The stress-energy tensor of a spinor field ($s = \frac{1}{2}$) in the spacetime $(V^{1,3}, g)$ is given by [3]:

$$T_{jk} = \frac{i}{2}\left(\overline{\psi}\gamma_{(j}\nabla_{k)}\psi - (\nabla_{(j}\overline{\psi})\gamma_{k)}\psi\right), \tag{6}$$

where $\gamma_j = \gamma_a t^a_j(x)$.

Taking into account equations (2), (3), (4) (5), and (6), we derive the Lie derivative of the stress-energy tensor:

$$\mathfrak{L}_{\xi}T_{jk} = \frac{\varphi}{2}\Bigg(T_{jk} - \frac{i}{4}\Big(\overline{\psi}\gamma_j t_{ka}\varphi_b\gamma^{ab}\psi + \overline{\psi}\gamma_k t_{ja}\varphi_b\gamma^{ab}$$
$$+\overline{\psi}t_{ka}\varphi_b\gamma^{ab}\gamma_j + \overline{\psi}t_{ja}\varphi_b\gamma^{ab}\gamma_k\psi\Big)\Bigg).$$

There exists a scalar quantity:

$$|A|^2 = A^i g_{ij} A^j = (\overline{\psi}\gamma^i\psi)\, g_{ij}\,(\overline{\psi}\gamma^j\psi),$$

where $A^i = \overline{\psi}\gamma^i\psi$ is the four-current of the spinor field ψ.

This scalar is invariant under conformal transformations:

$$\mathfrak{L}_{\xi}(|A|^2) = \mathfrak{L}_{\xi}\left(A^i g_{ij} A^j\right) = \mathfrak{L}_{\xi}\left(\overline{\psi}\gamma^i\psi \cdot g_{ij} \cdot \overline{\psi}\gamma^j\psi\right) = 0.$$

References

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