On boundary behavior of unclosed mappings with moduli inequality

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Let D be a domain in $\overline{{\Bbb R}^n}$ and let $b\in \partial D$. Then D has property P_1 at b if the following condition is satisfied: If E and F are connected subsets of D such that $b\in \overline{E}\cup\overline{F}$, then $M(\Gamma(E, F, D))=\infty$, where M denotes the (conformal) modulus of families of paths in ${\Bbb R}^n$ (see the definition below), and $\Gamma(E, F, D)$ is a family of paths joining E and F in D (see e.g. [1, Definition 17.5]). The following results hold.

Theorem A. *Suppose that $f:D\rightarrow D^{\,\prime}$ is a quasiconformal mapping and that D has property P_1 at $\in \partial D$. Then $C(f, b)$ contains at most one point at which $D^{\,\prime}$ is finitely connected (see* [1, Theorem 17.13]*).*

Theorem B. *Let $f:D\rightarrow {\Bbb R}^n$ be quasiregular mapping with $C(f, \partial D)\subset \partial f(D)$. If D is locally connected at a point $b\in \partial D$ and $D^{\,\prime}=f(D)$ is qc accessible at some point $y\in C(f, b)$, then $C(f, b)=\{y\}$ (see e.g.* [2, Theorem 4.2]*, cf.* [3, Theorem 4.2]*).*

We give some generalization of Theorems **A** and **B**. Recall some definitions. A Borel function $\rho:{\Bbb R}^n\,\rightarrow [0,\infty]$ is called *admissible* for the family Γ of paths γ in ${\Bbb R}^n,$ if the relation $\int\limits_{\gamma}\rho (x)\, |dx|\geqslant 1$ holds for all (locally rectifiable) paths $\gamma \in \Gamma.$ In this case, we write: $\rho \in {\rm adm} \,\Gamma.$ Let $p\geqslant 1,$ then *p-modulus* of Γ is defined by the equality $M_p(\Gamma)=\inf\limits_{\rho \in \,{\rm adm}\,\Gamma} \int\limits_{{\Bbb R}^n} \rho^p (x)\, dm(x)\,.$ Let $x_0\in {\Bbb R}^n,$ $0<r_1<r_2<\infty,$

$$S(x_0,r) = \{ x\,\in\,{\Bbb R}^n : |x-x_0|=r\}\,, \quad B(x_0, r)=\{x\in {\Bbb R}^n: |x-x_0|<r\} \tag{1}$$

and $A=A(x_0, r_1, r_2)=\{ x\,\in\,{\Bbb R}^n : r_1<|x-x_0|<r_2\}\,.$ Let $S_i=S(x_0, r_i),$ $i=1,2,$ where spheres $S(x_0, r_i)$ centered at x_0 of the radius r_i are defined in (1). Let $Q:{\Bbb R}^n\rightarrow {\Bbb R}$ be a Lebesgue measurable function satisfying the condition $Q(x)\equiv 0$ for $x\in{\Bbb R}^n\setminus D.$ Let $p\geqslant 1.$ Due to [4], a mapping $f:D\rightarrow \overline{{\Bbb R}^n}$ is called a *ring Q-mapping at the point $x_0\in \overline{D}\setminus \{\infty\}$ with respect to p-modulus*, if the condition

$$M_p(f(\Gamma(S_1, S_2, D)))\leqslant \int\limits_{A\cap D} Q(x)\cdot \eta^p (|x-x_0|)\, dm(x) \tag{2}$$

holds for some $r_0(x_0)>0,$ all $0<r_1<r_2< r_0$ and all Lebesgue measurable functions $\eta:(r_1, r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\, dr\geqslant 1\,. \tag{3}$$

Recall that a mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of each point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n.$ Later, in the extended space $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ we use the *spherical (chordal) metric* $h(x,y)=|\pi(x)-\pi(y)|$, where π is a stereographic projection $\overline{\mathbb{R}^n}$ onto the sphere $S^n(\frac{1}{2}e_{n+1},\frac{1}{2})$ in $\mathbb{R}^{n+1}$, namely,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,,\quad x\ne\infty\ne y$$

(see [1, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{\mathbb{R}^n}$ we understand relative to the chordal metric h in $\overline{\mathbb{R}^n}$. Given a mapping $f:D\rightarrow\mathbb{R}^n$, we denote $C(f,x):=\{y\in\overline{\mathbb{R}^n}:\exists\,x_k\in D:x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f,\partial D)=\bigcup\limits_{x\in\partial D}C(f,x)$. In what follows, $\operatorname{Int}A$ denotes the set of inner points of the set $A\subset\overline{\mathbb{R}^n}$. Recall that the set $U\subset\overline{\mathbb{R}^n}$ is neighborhood of the point z_0, if $z_0\in\operatorname{Int}A$. Due to [4], we say that a function $\varphi:D\rightarrow\mathbb{R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if $\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\varepsilon)}|\varphi(x)-\overline{\varphi}_{\varepsilon}|\ dm(x)<\infty$, where $\overline{\varphi}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\varepsilon)}\varphi(x)dm(x)$. Let $Q:\mathbb{R}^n\rightarrow[0,\infty]$ be a Lebesgue measurable function. We set $Q'(x)=\begin{cases}Q(x), & Q(x)\geqslant 1\,,\\ 1, & Q(x)<1\,.\end{cases}$ Denote by $q^{\,\prime}_{x_0}$ the mean value of $Q'(x)$ over the sphere $|x-x_0|=r$, that means,

$$q^{\,\prime}_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q'(x)\,d\mathcal{H}^{n-1}\,. \tag{4}$$

Note that, using the inversion $\psi(x)=\frac{x}{|x|^2}$, we may give the definition of FMO as well as the quantity in (4) for $x_0=\infty$. We say that the boundary ∂D of a domain D in $\mathbb{R}^n$, $n\geqslant 2$, is *strongly accessible at a point $x_0\in\partial D$ with respect to the p-modulus* if for each neighborhood U of x_0 there exist a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and $\delta>0$ such that $M_p(\Gamma(E,F,D))\geqslant\delta$ for each continuum F in D that intersects ∂U and ∂V.

Theorem 1. *([5]). Let $p\geqslant 1$, let D and D' be domains in $\mathbb{R}^n$, $n\geqslant 2$, $f:D\rightarrow D'$ be an open discrete mapping satisfying relations (2)–(3) at the point $b\in\partial D$ with respect to p-modulus, $f(D)=D'$. In addition, assume that 1) the set $E:=f^{\,-1}(C(f,\partial D))$ is nowhere dense in D and D is finitely connected on E, i.e., for any $z_0\in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components; 2) for any neighborhood U of b there is a neighborhood $V\subset U$ of b such that: 2a) $V\cap D$ is connected, 2b) $(V\cap D)\setminus E$ consists at most of m components, $1\leqslant m<\infty$, 3) $D'\setminus C(f,\partial D)$ consists of finite components, each of them has a strongly accessible boundary with respect to p-modulus. Suppose that at least one of the following conditions is satisfied: $4_1)$ a function Q has a finite mean oscillation at the point b; $4_2)$ $q_b(r)=O\left(\left[\log\frac{1}{r}\right]^{n-1}\right)$ as $r\rightarrow 0$; $4_3)$ the condition $\int\limits_{0}^{\delta(b)}\frac{dt}{t^{\frac{n-1}{p-1}}q_b^{\,\prime\,\frac{1}{p-1}}(t)}=\infty$ holds for some $\delta(b)>0$. Then f has a continuous extension to b.*

If the above is true for any point $b\in\partial D$, the mapping f has a continuous extension $\overline{f}:\overline{D}\rightarrow\overline{D'}$, moreover, $\overline{f}(\overline{D})=\overline{D'}$.

References

[1] Väisälä J. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229. Berlin etc., Springer–Verlag, 1971.

[2] Srebro U. Conformal capacity and quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I. Math.*, 529: 1–8, 1973.

[3] Vuorinen M. Exceptional sets and boundary behavior of quasiregular mappings in n-space. *Ann. Acad. Sci. Fenn. Ser. A 1. Math. Dissertationes*, 11: 1–44, 1976.

[4] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory.* Springer Science + Business Media, LLC : New York, 2009.

[5] Desyatka V., Sevost'yanov E. On boundary-non-preserving mappings with Poletsky inequality. *Canadian Mathematical Bulletin,* 2025. https://www.cambridge.org/core/journals/canadian-mathematical-bulletin/article/abs/ on-boundarynonpreserving-mappings-with-poletsky-inequality/F8994298FDE5B3D33D651F0A435F8990 .