Minkowski chirality

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In his ground-laying work [1], Grünbaum set up a general framework for quantifying the point (a)symmetry of convex bodies, i.e., compact convex sets with nonempty interior. Specifically, a measure of (a)symmetry is a similarity-invariant (or even affinely invariant) Hausdorff continuous function f that takes convex bodies to the unit interval with the property that $f(K) = 1$ if and only if K is point-symmetric.

In [1], some generalizations are discussed, for example quantifying (a)symmetry with respect to reflections across affine subspaces of dimension at least one. However, the author mentions lack of results in the literature in this direction. Different notions of chirality or axiality for quantifying the (a)symmetry of planar shapes with respect to reflections across straight lines have been investigated in the mathematical literature in the past decades. Asymmetry notions for planar convex bodies are also studied in mathematical chemistry , where polygons serve as abstractions of molecules and where chirality impacts chemical properties.

Our contribution is based on an extension of the notion of Minkowski asymmetry, which, for a convex body K, is defined as the smallest dilation factor $\lambda > 0$ such that K is a subset of a translated and dilated copy of $-K$, the mirror image of K upon reflection across the coordinate origin. We incorporate reflections across higher-dimensional (affine) subspaces by defining the *jth Minkowski chirality* $\alpha_j(K)$ as the smallest dilation factor $\lambda > 0$ such that the convex body $K \subset \mathbb{R}^n$ is a subset of a translated and dilated copy of $\Phi_U(K)$, where Φ_U denotes the *reflection* across the j-dimensional affine subspace $U \subset \mathbb{R}^n$ for $j \in \{0, \dots, n\}$. Note that the Minkowski asymmetry is $\alpha_0(K)$ in this terminology.

It is well-known that $\alpha_0(K) \in [1, n]$ for all convex bodies $K \subset \mathbb{R}^n$, with $\alpha_0(K) = 1$ if and only if K is point-symmetric, and $\alpha_0(K) = n$ if and only if K is a fulldimensional simplex, see [1].

Our main result for convex bodies in general dimensions extends the upper bound on the Minkowski asymmetry to all Minkowski chiralities $\alpha_j(K)$ for any $j \in \{0, \dots, n\}$.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a convex body and $j \in \{0, \dots, n\}$. Then*

$$1 \leq \alpha_j(K) \leq \min\{n, \frac{\alpha_0(K) + 1}{2}\sqrt{n}\},$$

with $\alpha_j(K) = 1$ if and only if there exists a j-dimensional affine subspace U such that $K = \Phi_U(K)$.

In fact, the upper bound in 1 can be strengthened and unified to

$$\alpha_j(K) \leq \sqrt{\alpha_0(K)n} \tag{1}$$

for any convex body $K \subset \mathbb{R}^n$ and $j \in \{0, \dots, n\}$. Since $\alpha_0(K) \leq n$ with $\alpha_0(K) = n$ solely for simplices, this result implies $\alpha_j(K) \leq n$ and in particular that only simplices might have jth Minkowski chirality n.

We recall that the *Banach–Mazur distance* between convex bodies $K, L \subset \mathbb{R}^n$ is defined by

$$d_{BM}(K, L) = \inf\{\lambda > 0 : t^1 + K \subset A(L) \subset t^2 + \lambda K,\ A \in GL(\mathbb{R}^n),\ t^1, t^2 \in \mathbb{R}^n\},$$

where $GL(\mathbb{R}^n)$ denotes the set of invertible real $n \times n$ matrices.

The inequality (1) is also consequential for the absolute upper bound on the jth Minkowski chirality. Any convex body K with Minkowski asymmetry $\alpha_0(K)$ near n is close to a simplex in the Banach-Mazur distance. Together with (1), this means that either the supremum of $\alpha_j(T)$ over all simplices

$T \subset \mathbb{R}^n$ equals n, or there exists some constant $c(n,j) < n$ such that any convex body $K \subset \mathbb{R}^n$ satisfies $\alpha_j(K) \le c(n,j)$. In other words, we can determine whether the inequality $\alpha_j(K) \le n$ is tight by checking only simplices.

Although this remains a challenging problem in general, we are able to solve it in the planar case for the first Minkowski chirality.

Theorem 2. *Let $K \subset \mathbb{R}^2$ be a triangle. Then the infimum in the definition of $\alpha_1(K)$ is attained at some affine subspace U of $\mathbb{R}^2$ that is necessarily*

(1) parallel to the bisector of one of the largest interior angles of K,
(2) parallel to the bisector of one of the smallest interior angles of K, or
(3) perpendicular to one of the longest edges of K.

Moreover, we have when $K \subset \mathbb{R}^2$ is a triangle

$$\alpha_1(K) = \left[1, \sqrt{2}\right), \tag{2}$$

with $\alpha_1(K) = 1$ precisely for isosceles triangles.

The question of how large $\alpha_j(K)$ can be for general n and j is still open, as even deciding whether the inequality $\alpha_j(K) \le n$ is actually tight appears to be difficult. Instead, we focus on a special class of convex bodies and answer the first question for planar point-symmetric convex bodies: the upper bound from 1 becomes $\sqrt{2}$ in this case, and the following two theorems show that this bound is reached precisely by a specific family of parallelograms.

The second theorem uses the *John ellipsoid* $\mathcal{E}_J(K)$ of a convex body $K \subset \mathbb{R}^n$, which is the unique volume-maximal ellipsoid contained in K.

Theorem 3. *Let $K \subset \mathbb{R}^2$ be a point-symmetric convex body with $d_{BM}(K,P) \ge 1+\epsilon$ for a parallelogram $P \subset \mathbb{R}^2$ and some $\epsilon > 0$. Then*

$$\alpha_1(K) < \sqrt{2}\left(1 - \frac{\epsilon}{10}\right).$$

Theorem 4. *Let $K \subset \mathbb{R}^2$ be a parallelogram. Then the infimum in the definition of $\alpha_1(K)$ is attained at some affine subspace U of $\mathbb{R}^2$ that is necessarily parallel to*

(1) the bisector of an angle formed by consecutive edges of K,
(2) the bisector of an angle formed by the diagonals of K, or
(3) a principal axis of the John ellipse $\mathcal{E}_J(K)(K)$ of K.

Moreover, we have when $K \subset \mathbb{R}^2$ is a parallelogram

$$\alpha_1(K) = \left[1, \sqrt{2}\right], \tag{3}$$

with $\alpha_1(K) = 1$ precisely for rectangles and rhombuses. Moreover, $\alpha_1(K) = \sqrt{2}$ if and only if the angles between the diagonals coincide with the interior angles and the ratio between the lengths of the longer edges and the shorter edges is at least $\sqrt{2}$.

References

[1] Branko Grünbaum. Measures of symmetry for convex sets. *Proc. Sympos. Pure Math., Vol. VII,*: 233–270, 1963.