Forbidden Four Cycle in Diametrical Graphs and Embedding in Stars

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The ultrametric spaces generated by arbitrary nonnegative vertex labelings on both finite and infinite trees were first considered in [2] and studied in [5, 4]. The simplest types of infinite trees are rays and star graphs. The totally bounded ultrametric spaces generated by labeled almost rays have been characterized in [7]. Furthermore, paper [6] contains a purely metric characterization of ultrametric spaces generated by labeled star graphs.

Our main purpose is to give an answer to the following problem.

Problem 1. Let (X, d) be an ultrametric space. Find conditions under which (X, d) admits an isometric embedding in an ultrametric space generated by labeled star graph.

Here and in what follows, by *labeled* star graph $S(l)$ we will mean a star graph S equipped with a labeling $l\colon V(S) \to \mathbb{R}^+$, where $V(S)$ is the vertex set of the star graph S.

Let $S(l)$ be a labeled star graph. As in [2] we define a mapping $d_l\colon V(S) \times V(S) \to \mathbb{R}^+$ by

$$d_l(u, v) = \begin{cases} 0, & \text{if } u = v, \\ \max\limits_{w \in V(P)} l(w), & \text{otherwise,} \end{cases}$$

where P is the path joining u and v in S. Let (Y, ρ) be an ultrametric space. We say that (Y, ρ) is generated by labeled star graph $S(l)$ if Y is the vertex set of S and the equality $\rho = d_l$ holds.

We will also use the concept of diametrical graph introduced in [1]. The next definition is a modification of Definition 2.1 from [8].

Definition 2. Let (X, d) be an ultrametric space with $\operatorname{card} X \geq 2$. A graph G is called the diametrical graph of (X, d) if X is the vertex set of G and points $x, y \in X$ are adjacent in G if and only if

$$d(x, y) = \sup\{d(u, v)\colon u,\, v \in X\}.$$

The following theorem gives a solution of Problem 1.

Theorem 3. *Let (X, d) be an infinite ultrametric space. Then the following statements are equivalent:*

(i) *There is (Y, ρ) such that (Y, ρ) is generated by labeled star graph and (X, d) is isometric to a subspace of (Y, ρ).*

(ii) *(X, d) contains no four-point subspace with diametrical graph isomorphic to the cycle C_4.*

If (X, d) is a compact ultrametric space, then Theorem 3 follows from Theorem 5.2 of [3].

References

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