Constant mean curvature surfaces with harmonic Gauss maps in three-dimensional Lie groups

Eugene Petrov
(V.N. Karazin Kharkiv National University, Kharkiv, Ukraine)
E-mail: `petrov@karazin.ua`

Let G be a $(n+1)$-dimensional Lie group with a left invariant metric. For an (immersed) hypersurface M in G define the Gauss map Φ of M by

$$\Phi\colon M \to \mathbb{R}P^n; \Phi(p) = dL_{p^{-1}}(N_pM).$$

Here a point p is identified with its image under the immersion, N_pM is a normal space of N at p, and $dL_{p^{-1}}$ is the differential of the left translation in G. If M is orientable we can consider also the orientable Gauss map whose target space is S^n. It was shown in [1] that if the metric of G is biinvariant then Φ is harmonic if and only if M is of constant mean curvature (CMC). This is a generalization of a classical Euclidean result of [4]. In particular, for $n = 2$ biinvariant metrics exist on the simply-connected Lie groups $\mathbb{R}^3$ and $\mathbb{S}^3$ and are their usual metrics of constant curvature. It appears that for many other classes of left invariant metrics the equivalence between the harmonicity of Φ and CMC does not take place. For example, each CMC hypersurface with the harmonic Gauss map in the $(2m+1)$-dimensional Heisenberg group is locally a vertical cylinder ([3]).

Using a well-known description from [2] of left invariant metrics on a three-dimensional Lie group G, we derive criteria of the harmonicity of Φ for a general such metric. This allows us to prove the following:

Theorem 1. *Let a left invariant metric on a connected three-dimensional unimodular Lie group G be right invariant with respect to a one-dimensional subgroup $H \subset G$, but not biinvariant, and let M be a connected surface in G. Then from any two of the following claims the third follows:*

(1) M is CMC;
(2) the Gauss map of M is harmonic;
(3) M is either everywhere orthogonal to the one-dimensional foliation generated by H (is horizontal) or is composed of leaves of this foliation (is vertical).

This applies to all left invariant metrics on the 3-dimensional Heisenberg group, some metrics on the groups E(2) (of orientation-preserving Euclidean plane isometries), $\mathrm{SL}(2,\mathbb{R})$ and their universal coverings, and to some non-biinvariant metrics on SU(2) and its universal covering $\mathbb{S}^3$. It allows us to give explicit descriptions of CMC surfaces with harmonic Gauss maps for some model metrics on these groups. We also consider some examples of metrics that neither are biinvariant nor satisfy the conditions of the theorem 1 (in particular, any left invariant metric on the Lie group Sol is like that).

We also use the Gauss map harmonicity criteria for non-unimodular groups to prove the following:

Theorem 2. *A complete connected surface in the hyperbolic space $\mathbb{H}^3$ is CMC with the harmonic Gauss map (in the Lie group sense) if and only if it is a horosphere parallel to the sphere at infinity.*

Here the Lie group structure on the half-space $(z > 0)$ model of $\mathbb{H}^3$ with the usual metric $\frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ corresponds to the orthonormal frame of left-invariant fields $X_1 = z\frac{\partial}{\partial x}$, $X_2 = z\frac{\partial}{\partial y}$, $X_3 = z\frac{\partial}{\partial z}$, and the horospheres are thus of the form $z = z_0$.

These results were partially obtained in a joint work with Iryna Savchuk.

References

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