On controllability problems for the heat equation on a half-plane controlled by the Neumann boundary condition with a point-wise control

Larissa Fardigola
(B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine, V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine)
E-mail: fardigola@ilt.kharkov.ua

Kateryna Khalina
(B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine)
E-mail: khalina@ilt.kharkov.ua

Consider the following control system on a half-plane

$$w_t = \Delta w, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1}$$

$$w_{x_1}\big(0,(\cdot)_{[2]},t\big) = \delta_{[2]}u(t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{2}$$

$$w\big((\cdot)_{[1]},(\cdot)_{[2]},0\big) = w^0, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R}, \tag{3}$$

where $\mathbb{R}_+ = (0,\infty)$, $T > 0$, $u \in L^\infty(0,T)$ is a control, $\delta_{[m]}$ is the Dirac distribution with respect to x_m, $m = 1,2$, $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. The subscripts [1] and [2] associate with the variable numbers, e.g., $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. This control system is considered in spaces of Sobolev type. We treat equality (2) as the value of the distribution w_{x_1} on the line $x_1 = 0$.

A point-wise control is a mathematical model of a source supported in a domain of very small size with respect to the whole domain. That is why studying control problems under a point-wise control is an important issue in control theory.

The controllability problems for the heat equation on a half-plane controlled by the Neumann boundary condition with a point-wise control is studied. These problems for the heat equation on a half-plane controlled by the Dirichlet boundary condition with a point-wise control were studied in [1].

Let $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. By $\mathcal{R}_T\left(w^0\right)$, denote the set of all states $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ for which there exists a control $u \in L^\infty(0,T)$ such that there exists a unique solution w to system (1)–(3) such that $w\big((\cdot)_{[1]},(\cdot)_{[2]},T\big) = w^T$.

Definition 1. A state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is said to be controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time $T > 0$ if $w^T \in \mathcal{R}_T\left(w^0\right)$.

Definition 2. A state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is said to be approximately controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time $T > 0$ if $w^T \in \overline{\mathcal{R}_T\left(W^0\right)}$, where the closure is considered in the space $L^2(\mathbb{R}_+ \times \mathbb{R})$.

For control system (1)–(3), the set $\mathcal{R}_T(0) \subset L^2(\mathbb{R}_+ \times \mathbb{R})$ of its states reachable from 0 (i.e. the set which is formed by the end states $w(\cdot,T)$ of this system when controls $u \in L^\infty(0,T)$) and the set $\mathcal{R}_T^L(0) \subset \mathcal{R}_T(0) \subset L^2(\mathbb{R}_+ \times \mathbb{R})$ of its states reachable from 0 by using the controls $u \in L^\infty(0,T)$ satisfying the restriction $\|u\|_{L^\infty(0,T)} \le L$ (where $L > 0$ is a given constant) are studied to investigate the (approximate) controllavility properties. It is established that a function $f \in \mathcal{R}_T(0)$ can be represented in the form $f(x) = g\big(|x|^2\big)$ a.e. in $\mathbb{R}_+ \times \mathbb{R}$ where $g \in L^2(0,+\infty)$. In fact, the problem dealing with functions from $L^2(\mathbb{R}_+\times\mathbb{R})$ is reduced to a problem dealing with functions from $L^2(0,+\infty)$. To this aid, operators Ψ and Φ are introduced and studied (see below).

If $f\in L^2(\mathbb{R}_+\times\mathbb{R})$ and $f(x)=g(|x|^2)$, $x\in\mathbb{R}_+\times\mathbb{R}$, for some g defined on $\mathbb{R}_+$, then $g\in L^2(\mathbb{R}_+)$ and $\|f\|_{L^2(\mathbb{R}_+\times\mathbb{R})}=\sqrt{\frac{\pi}{2}}\|g\|_{L^2(\mathbb{R}_+)}$ holds; and vice versa: if $g\in L^2(\mathbb{R}_+)$, then for $f(x)=g(|x|^2)$, $x\in\mathbb{R}_+\times\mathbb{R}$, we have $f\in L^2(\mathbb{R}_+\times\mathbb{R})$. Taking this into account, we can introduce the space

$$\mathcal{H}=\left\{f\in L^2(\mathbb{R}_+\times\mathbb{R})\;\middle|\;\exists g\in L^2(\mathbb{R}_+)\quad f(x)=g(|x|^2)\text{ a.e. on }\mathbb{R}_+\times\mathbb{R}\right\} \tag{4}$$

and the operator $\Psi:\mathcal{H}\to L^2(\mathbb{R}_+)$ with the domain $D\big(\Psi\big)=\mathcal{H}$ for which

$$\Psi f=g\Leftrightarrow\big(f(x)=g(|x|^2)\text{ a.e. on }\mathbb{R}_+\times\mathbb{R},\quad f\in D\big(\Psi\big)=\mathcal{H}.$$

One can see that Ψ is invertible, $\Psi^{-1}:L^2(\mathbb{R}_+)\to\mathcal{H}$, and $\big(\Psi^{-1}g\big)(x)=g(|x|^2)$, $x\in\mathbb{R}_+\times\mathbb{R}$ for $g\in D\big(\Psi^{-1}\big)=L^2(\mathbb{R}_+)$.

Thus, Ψ is an isomorphism of $\mathcal{H}$ and $L^2(\mathbb{R}_+)$, and $\|\Psi\|=\sqrt{\frac{2}{\pi}}$. Moreover, $\mathcal{H}$ is a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2(R_+\times\mathbb{R})}$ and $2\langle f,h\rangle_{L^2(R_+\times\mathbb{R})}=\pi\langle\Psi f,\Psi h\rangle_{L^2(\mathbb{R}_+)}$, $f\in\mathcal{H}$, $h\in\mathcal{H}$.

Let us introduce the operator $\Phi:L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)$ with the domain $D(\Phi)=L^2(\mathbb{R}_+)$ by the rule

$$\big(\Phi g\big)(\rho)=\frac{1}{2}\lim_{N\to\infty}\int_0^N g(r)J_0\big(\sqrt{r\rho}\big)\,dr,\quad \rho\in\mathbb{R}_+,\ g\in L^2(\mathbb{R}_+),$$

where J_0 is the Bessel function of order 0. We prove that Φ is invertible and $\Phi^{-1}=\Phi$, in particular, Φ is an isometric isomorphism of $L^2(\mathbb{R}_+)$. Note that the transform providing by the operator Φ is a modification of the well-known Hankel transform of order 0.

The operators Ψ and Φ are key tools of this work, which allow to obtain the following main results:

- **(a)** some properties of the set $\mathcal{R}_T(0)$, in particular, $\overline{\mathcal{R}_T(0)}=\mathcal{H}$;
- **(b)** some properties of the set $\mathcal{R}_T^L(0)$;
- **(c)** necessary and sufficient conditions for controllability in a given time under the control bounded by a given constant;
- **(d)** sufficient conditions for approximate controllability in a given time under the control bounded by a given constant;
- **(e)** necessary and sufficient conditions for approximate controllability in a given time, in particular, the origin can be driven to a given state $w^T\in L^2(R_+\times\mathbb{R})$ in a given time T iff $w^T\in\mathcal{H}$;
- **(f)** the lack of controllability to the origin.

Results **(c)** and **(d)** are obtained from **(b)**, and result **(e)** follows from **(a)**. The method of obtaining result **(f)** is very similar to that in [3]. The results are illustrated by examples.

The main results of the present paper are rather similar to those of [1]. However, the methods of obtaining them are essentially different in these two papers. Roughly speaking, we deal with the two-dimensional case studying reachability sets and constructing the solutions to controllability and approximate controllability problems in [1], but reducing the two-dimensional reachability sets to the one-dimensional ones, we deal with the one-dimensional case studying these problems and constructing their solutions in the present paper. In addition, the methods used to study the one-dimensional reachability sets in this paper principally differ from those used for two-dimensional sets in [1]. Moreover, some results of the present work have not analogues in [1]. Most of the obtained results were published in [2].

References

[1] L. Fardigola and K. Khalina. Controllability problems for the heat equation on a half-plane controlled by the Dirichlet boundary condition with a point-wise control. *J. Math. Phys., Anal., Geom.*, 18(1) : 75–104, 2022.

[2] L. Fardigola and K. Khalina. Controllability problems for the heat equation on a half-plane controlled by the Neumann boundary condition with a point-wise control. *J. Math. Phys., Anal., Geom.*, 21(1) : 23–55, 2025.

[3] S. Micu and E. Zuazua. On the lack of null controllability of the heat equation on the half-space, *Port. Math. (N.S.)* 58(1) : 1–24, 2001.