Normal forms of Morse-Bott functions without saddles on compact oriented surfaces

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Let M be a smooth compact and oriented surface, and denote by P a real line $\mathbb{R}$ or a circle S^1. Denote by $\mathcal{F}^0(M,P)$ a class of Morse-Bott functions without saddles on M with the value in P. This class of functions naturally arises in the study of homotopy type of stabilizers of Morse-Bott functions on surfaces with respect to the action of the group of diffeomorphisms by pre-composition, see details in [1]. It is known that this class is non-empty if M is diffeomorphic to one of the following list: a cylinder $S^1 \times [0,1]$, a disk D^2, a sphere S^2, a torus T^2.

There are some trivial examples of functions from $\mathcal{F}^0(M,P)$ that are easy to write by hand:

Example 1. Let $f_0 : M_0 \to P$ be a smooth function from $\mathcal{F}^0$

- (1_0) $M_0 = S^1 \times [0,1] = \{(z,s) \,|\, z \in \mathbb{C}, |z| = 1,\, 0 \le s \le 1\}$ is a unit cylinder, and $f_0 : S^1 \times [0,1] \to \mathbb{R}$ is given by $f_0(z,s) = s$,
- (2_0) $M_0 = D^2 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \le 1\}$ is a unit 2-disk, and $f_0 : D^2 \to \mathbb{R}$ is given by $f_0(x,y) = x^2 + y^2$,
- (3_0) $M_0 = S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ is a unit sphere, and $f_0 : S^2 \to \mathbb{R}$ is given by $f_0(x,y,z) = z$,
- (4_0) $M_0 = T^2 = \{(w,z) \in \mathbb{C}^2 \,|\, |z| = |w| = 1\}$ is a unit 2-torus, and $f_0 : T^2 \to S^1$ is given by $f_0(w,z) = z$.

Note that these functions do not have critical circles. We will call them **prime functions**.

Our main result is the following theorem, see [2].

Theorem 2. *A function $f \in \mathcal{F}^0(M,P)$ admits the following decomposition*

$$f = \varkappa \circ f_0 \circ h^{-1} \tag{1}$$

where $h : M_0 \to M$ is a diffeomorphism, $f_0 \in \mathcal{F}^0(M_0,P)$ is a prime function, and a smooth function $\varkappa : f_0(M_0) \to P$ which satisfies the following conditions:

- (A) *$\varkappa$ has the only finite number of non-degenerated critical points,*
- (B) *$\varkappa$ does not have critical points at $f_0(\Sigma_{f_0})$ and $f_0(\partial M)$,*

where Σ_{f_0} is the set of critical points of f_0. A factorization (1) *is not unique and depends on the choice of h. In particular, if f has no critical circles, then $\varkappa$ is a diffeomorphism.*

References

[1] Bohdan Feshchenko. Homotopy type of stabilizers of circle-valued functions with non-isolated singularities on surfaces. *arXiv:2305.08255*, 9p., 2023

[2] Bohdan Feshchenko. Normal forms of functions with degenerate singularities on surfaces equipped with semi-free circle actions. *arXiv:2412.18944*, 18p., 2024.