On one geometric application of the Sturm–Hurwitz theorem

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For a smooth closed curve γ with curvatures $k_1 > 0$, $k_2 > 0$, and k_3 in the four-dimensional Euclidean space $\mathbb{E}^4$, we explore the well-defined integral quantity

$$J(\gamma) = \oint_{\gamma} \sqrt{k_1^2 + k_2^2 + k_3^2}ds,$$

which is invariant under rigid motions and dilatations in $\mathbb{E}^4$. We address the problem of determining the sharp lower bound for $J(\gamma)$, see [1].

Clearly, $J(\gamma) \geq 2\pi$ in view of the classical Fenchel inequality $\oint_{\gamma} k_1 ds \geq 2\pi$. However, if γ has constant curvatures then the stronger estimate $J(\gamma) \geq 2\sqrt{5}\pi$ holds true, and this estimate is sharp, see [2].

We conjecture that the same inequality $J(\gamma) \geq 2\sqrt{5}\pi$ holds true in the general situation as well. At the moment, the conjecture remains still unproven.

We consider the limit situation where γ evolves smoothly into a unit circle. Specifically, we introduce a smooth family of closed curves $\{\gamma_\varepsilon\}_{\varepsilon \geq 0}$ in $\mathbb{E}^4$ represented by the position vector $x(t) = (\cos t, \sin t, \varepsilon w_1(t), \varepsilon w_2(t))$, where $w_1(t)$, $w_2(t)$ are smooth 2π-periodic functions. This family is viewed as a perturbation of the unit circle γ_0.

Clearly, all the geometric features of γ_ε are determined by the vector-function $w(t) = (w_1(t), w_2(t))$. In particular, γ_ε with $\varepsilon > 0$ satisfy $k_1 > 0$ and $k_2 > 0$ if and only if $w(t)$ satisfies $w'' + w \neq 0$. In this generic case, the value of $J(\gamma_\varepsilon)$ is well-defined for $\varepsilon > 0$, and one can explore its limit value as $\varepsilon \to 0$.

We provide a geometrically meaningful description for the value of $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon)$ in terms of the planar curve Γ represented by $p = w'' + w$, and then we demonstrate, as the main result, that this limit value cannot be less that $2\sqrt{5}\pi$,

$$\lim_{\varepsilon \to 0} J(\gamma_\varepsilon) \geq 2\sqrt{5}\pi,$$

for any choice of $w(t)$. Moreover, the inequality is proved to be sharp in the sense that one can chose $w(t)$ with $w'' + w \neq 0$ so that $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon) = 2\sqrt{5}\pi$. Thus, the proved statement provides novel non-trivial arguments supporting the conjecture under consideration.

The proof of the main result is based on the use of the Sturm–Hurwitz theorem regarding the number of zeroes of trigonometric polynomials / Fourier series, see [3], [4]. We apply this celebrated theorem of the mathematical analysis to estimate a specific tangency complexity of the planar curve Γ leading to the desired lower bound for $\lim_{\varepsilon \to 0} J(\gamma_\varepsilon)$.

References

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