Harmonic number series related to specific generating functions

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The central binomial coefficients are defined for integer $n \geq 0$ by $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. These numbers are closely related to the well-known Catalan numbers, given by $C_n = \frac{1}{n+1}\binom{2n}{n}$. The harmonic numbers of order m and the odd harmonic numbers of order m are defined respectively by $H_n^{(m)} = \sum\limits_{k=1}^{n} \frac{1}{k^m}$ and $O_n^{(m)} = \sum\limits_{k=1}^{n} \frac{1}{(2k-1)^m}$ with $H_0^{(m)} = O_0^{(m)} = 0$. The cases $H_n^{(1)} = H_n$ and $O_n^{(1)} = O_n$ correspond to the ordinary harmonic and odd harmonic numbers, respectively.

In this note, we present several infinite series involving central binomial coefficients, Catalan numbers, harmonic numbers, products of harmonic numbers, and mixed products with odd harmonic numbers. The method used to derive these expressions relies on integration—a classical technique that has recently gained renewed attention in the literature [1–3], among others.

Theorem 1. *We have*

$$\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_n}{n} = \frac{\pi^2}{3}, \qquad \sum_{n=0}^{\infty} \frac{C_n H_{n+1}}{2^{2n}} = 4,$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_{n+r}}{n+r} = \frac{2^{2r+1}}{\binom{2r}{r}} \frac{O_r}{r}, \quad r \neq 0,$$

and more generally, for $s - \frac{1}{2} \notin \mathbb{Z}_{<0}$, $r \neq 0$, and $r + s - \frac{1}{2} \notin \mathbb{Z}_{<0}$,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}} \frac{H_{n+r+s} - H_s}{\binom{n+r+s}{s+1}} = \frac{2(s+1)}{2s+1} \frac{H_{r+s-1/2} - H_{s-1/2}}{\binom{r+s-1/2}{r-1}}.$$

Theorem 2. *We have*

$$\sum_{n=0}^{\infty} \frac{O_{n+1}}{(n+1)(2n+1)} = \frac{\pi^2}{6}, \qquad \sum_{n=0}^{\infty} \frac{C_n O_{n+3}}{2^{2n}(2n+5)} = \frac{8}{9} + \frac{\pi}{32} - \frac{\pi \ln 2}{4},$$

$$\sum_{n=0}^{\infty} \frac{C_n H_{n+r+1}}{2^{2n+1}(n+r+1)} = \frac{H_r}{r} - \frac{2^{2r}}{4r^2-1} \frac{O_{r+1}}{\binom{2(r-1)}{r-1}},$$

and more generally, for $s, r \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and $r + s - \frac{1}{2} \notin \mathbb{Z}_{<0}$,

$$\sum_{n=0}^{\infty} \frac{C_n}{2^{2n+1}} \frac{H_{n+r+s} - H_{s-1}}{\binom{n+r+s}{s}} = \frac{H_{r+s-1} - H_{s-1}}{\binom{r+s-1}{s}} - \frac{2s}{2s+1} \frac{H_{r+s-1/2} - H_{s-1/2}}{\binom{r+s-1/2}{r-1}}.$$

Theorem 3. *We have*

$$\sum_{n=1}^{\infty}\frac{H_nH_{n+1}}{n\,(n+1)}=\frac{\pi^2}{6}+2\zeta(3),\qquad \sum_{n=1}^{\infty}\frac{H_nO_n}{(2n-3)(2n-1)}=\frac{\pi^2}{36}-\frac{1}{6}+\frac{1}{2}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{H_nH_{n+s+1}}{(n+s)(n+s+1)}=\frac{H_s}{s}+\frac{H_s^2+H_s^{(2)}}{s},\quad s\neq 0,$$

where $\zeta(s)=\sum\limits_{n=1}^{\infty}\frac{1}{n^s}$, $(\Re(s)>1)$ *is the Riemann zeta function. More generally, for* $0\leq r\in\mathbb{C}\setminus\mathbb{Z}_{<0}$, $s\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$,

$$\sum_{n=1}^{\infty}\frac{H_nH_{n+r+s+1}}{\binom{n+r+s+1}{r+2}}=\frac{r+2}{s\binom{r+s}{r}}H_{r+1}(H_{r+s}-H_r)+\frac{r+2}{s\binom{r+s}{r}}\Big((H_{r+s}-H_r)^2+H_{r+s}^{(2)}-H_r^{(2)}\Big).$$

Theorem 4. *We have*

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{2^{2n}}\frac{O_n}{n}=\sum_{n=1}^{\infty}\frac{2^{2n}H_n}{n(n+1)\binom{2(n+1)}{n+1}},\qquad \sum_{n=1}^{\infty}\frac{C_nO_n}{2^{2n}}=\sum_{n=1}^{\infty}\frac{2^{2(n+1)}H_n}{(n+1)(n+2)\binom{2(n+2)}{n+2}},$$

and more generally, if $r\in\mathbb{Z}_{\geq 0}$, *then*

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}O_n}{2^{2n}\,(n+r)}=\sum_{n=1}^{\infty}\frac{2^{2(n+r)}H_n}{(n+r)(n+r+1)\binom{2(n+r+1)}{n+r+1}}.$$

Theorem 5. *For all* $x\in[-1/4,1/4)$,

$$\begin{aligned}2\sqrt{1-4x}&\sum_{n=1}^{\infty}\binom{2n}{n}H_nO_nx^n\\&=\frac{\pi^2}{2}+\mathrm{Li}_2(1-4x)-4\mathrm{Li}_2\big(\sqrt{1-4x}\big)+\ln(1-4x)\left(\ln\left(\frac{1-4x}{|x|}\right)-\frac{1-\operatorname{sgn}x}{2}\pi i\right),\end{aligned}$$

where $\mathrm{Li}_2(x)$ *denotes the dilogarithm function, defined by* $\mathrm{Li}_2(x)=\sum\limits_{k=1}^{\infty}\frac{x^k}{k^2}$, $|x|\leq 1$.

Theorem 6. *We have*

$$\sum_{n=1}^{\infty}\frac{C_nH_nO_n}{2^{2n}}=\frac{\pi^2}{2}+4\ln 2,\qquad \sum_{n=1}^{\infty}\frac{C_nH_nO_n}{2^{2n}\,(n+2)}=-\frac{14}{9}+\frac{\pi^2}{6}+\frac{4}{9}\ln 2,$$

$$\sum_{n=1}^{\infty}\frac{H_nO_n}{(2n+1)\,(2n+3)}=\frac{1}{2}+\frac{\pi^2}{24}-\frac{1}{2}\ln 2,$$

and, more generally, for $r\in\mathbb{C}\setminus\mathbb{Z}_{\leq 0}$,

$$\sum_{n=1}^{\infty}\frac{\binom{2n}{n}}{\binom{n+(r+1)/2}{n}}\frac{H_nO_n}{2^{2n}}=\frac{2(r+1)}{r^2}\left(H_r-H_{r/2}-\frac{r}{4}H_{r/2}^{(2)}+\frac{\pi^2}{24}r-\ln 2+\frac{2}{r}\right).$$

References

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