On non-topologizable semigroups of the bicyclic monoid

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In this paper we shall follow the semigroup terminology of [1, 2, 4, 5].

Throughout these abstract we always assume that all topological spaces involved are Hausdorff — unless explicitly stated otherwise.

Definition 1. Let X, Y and Z be topological spaces. A map $f\colon X\times Y\to Z$, $(x,y)\mapsto f(x,y)$, is called

- (i) *right* [*left*] *continuous* if it is continuous in the right [left] variable; i.e., for every fixed $x_0\in X$ [$y_0\in Y$] the map $Y\to Z$, $y\mapsto f(x_0,y)$ [$X\to Z$, $x\mapsto f(x,y_0)$] is continuous;
- (ii) *separately continuous* if it is both left and right continuous;
- (iii) *jointly continuous* if it is continuous as a map between the product space $X\times Y$ and the space Z.

Definition 2. Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation) $\mu\colon S\times S\to S$, $(x,y)\mapsto\mu(x,y)=xy$. Then the pair (S,μ) is called

- (i) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s\colon S\to S$, $x\mapsto sx$, are continuous maps, $s\in S$;
- (ii) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s\colon S\to S$, $x\mapsto xs$, are continuous maps, $s\in S$;
- (iii) a *semitopological semigroup* if the map μ is separately continuous;
- (iv) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- an *left-continuous* topology if (S,τ) is a left topological semigroup;
- an *right-continuous* topology if (S,τ) is a right topological semigroup.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^kp^l\cdot q^mp^n=\begin{cases} q^{k-l+m}p^n, & \text{if } l<m;\\ q^kp^n, & \text{if } l=m;\\ q^kp^{l-m+n}, & \text{if } l>m.\end{cases}$$

We define the following subsets of the bicyclic monoid

$$\mathscr{C}_+(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\leqslant j\right\}\quad\text{and}\quad \mathscr{C}_-(p,q)=\left\{q^ip^j\in\mathscr{C}(p,q)\colon i\geqslant j\right\}.$$

For an arbitrary non-negative integer k we define

$$\mathscr{C}_{+k}(a,b)=\left\{b^ia^{i+s}\in\mathscr{C}_+(a,b)\colon s\geqslant k, s\in\omega\right\}.$$

Fix an arbitrary infinite subset X of ω. Latter we shall assume that $X=\{x_i\colon i\in\omega\}$ where $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. Put

$$\mathscr{C}_{+k}^{X}(a,b)=\mathscr{C}_{+k}(a,b)\cup\left\{b^{x_i}a^{x_i}\in\mathscr{C}_{+}(a,b)\colon i\in\omega\right\}.$$

The set $\mathscr{C}_{+k}^{X}(a,b)$ is a subsemigroup of $\mathscr{C}_{+}(a,b)$. By dual way we define the subsemigroup $\mathscr{C}_{-k}^{X}(a,b)$ of $\mathscr{C}_{-}(a,b)$.

Theorem 3. *The monoid $\mathscr{C}_{+}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every left-continuous Hausdorff topology on $\mathscr{C}_{+k}^{X}(a,b)$ is discrete.*

Theorem 4. *The monoid $\mathscr{C}_{-}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every right-continuous Hausdorff topology on $\mathscr{C}_{+k}^{X}(a,b)$ is discrete.*

Proposition 5. *The monoid $\mathscr{C}_{+}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, and there exists a Hausdorff topology τ on $\mathscr{C}_{+k}^{X}(a,b)$ such that the semigroup operation on $(\mathscr{C}_{+k}^{X}(a,b),\tau)$ is left-continuous but it is not right-continuous.*

Proposition 6. *The monoid $\mathscr{C}_{-}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, and there exists a Hausdorff topology τ on $\mathscr{C}_{-k}^{X}(a,b)$ such that the semigroup operation on $(\mathscr{C}_{-k}^{X}(a,b),\tau)$ the semigroup operation is left-continuous but it is not right-continuous.*

The set $\mathscr{C}_{\mathbb{Z}}=\mathbb{Z}\times\mathbb{Z}$ with the following semigroup operation

$$(k,l)\cdot(m,n)=\begin{cases}(k-l+m,n), & \text{if } l<m;\\ (k,n), & \text{if } l=m;\\ (k,l-m+n), & \text{if } l>m.\end{cases}$$

is called the *extended bicyclic semigroup* [6]. Every Hausdorff shift-continuous topology on the semigroup $\mathscr{C}_{\mathbb{Z}}$ is discrete [3]. We construct continuum subsemigroups S of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ such that the statements of Theorem 3 and Propositions 5 (Theorem 4 and Propositions 6) hold for S and every element of S is not maximal with the respect to the induced natural partial order from the inverse semigroup $\mathscr{C}_{\mathbb{Z}}$.

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