Stability of minimal surfaces in three-dimensional sub-Riemannian Lie groups

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A *sub-Riemannian manifold* is a smooth manifold M together with a completely non-integrable smooth distribution $\mathcal{H}$ on M (it is called a *horizontal distribution*) and a smooth field of Euclidean scalar products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on $\mathcal{H}$ (it is called a *sub-Riemannian metric*). In particular, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ can be constructed as a restriction of some Riemannian metrics $\langle\cdot,\cdot\rangle$ on M to $\mathcal{H}$. Here we will assume that all sub-Riemannian structures are of this form. Let Σ be a smooth oriented surface in a three-dimensional sub-Riemannian manifold M. If N_h is the orthogonal projection of the unit normal field N of Σ (in the Riemannian sense) onto $\mathcal{H}$ and $d\Sigma$ is the Riemannian area form of Σ, then the *sub-Riemannian area* of a domain $D \subset \Sigma$ is defined as $A(D) = \int\limits_D |N_h|\, d\Sigma$. The *normal variation* of the surface Σ defined by a smooth function u is the map $\varphi\colon \Sigma \times I \to M\colon \varphi_s(p) = \exp_p(su(p)N(p))$, where I is an open neighborhood of 0 in $\mathbb{R}$ and $\exp_p$ is the Riemannian exponential map in p. Denote $A(s) = \int\limits_{\Sigma_s} |N_h|\, d\Sigma_s$, where $\Sigma_s = \varphi_s(\Sigma)$. Then $A'(0)$ is called the *first (normal) area variation* defined by φ, and $A''(0)$ is called the *second* one. A surface Σ is called *minimal* if $A'(0) = 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$, where $\Sigma_0 = \{p \in \Sigma \mid N_h(p) = 0\}$ is the *singular set* of Σ. A minimal surface Σ is called *stable* if $A''(0) \geq 0$ for any normal variations with compact support in $\Sigma \setminus \Sigma_0$. We will call a surface Σ in a three-dimensional sub-Riemannian manifold *vertical* if $T_p\Sigma \perp \mathcal{H}_p$ for each $p \in \Sigma$. In particular, for such surfaces $N_h = N$ and $\Sigma_0 = \varnothing$.

In [1] we proved that a vertical surface Σ is minimal in the sub-Riemannian sense if and only if it is minimal in the Riemannian sense and derived the following second variation formula:

$$A''(0) = \int_\Sigma -\left(X(u) - \langle \nabla_N X, N\rangle u\right)^2 + |\nabla_\Sigma u|^2 - \left(\mathrm{Ric}\,(N,N) + |B|^2\right) u^2\, d\Sigma,$$

where ∇ and Ric are the Riemannian connection and the Ricci tensor of M respectively, X is the unit normal vector field of $\mathcal{H}$ (which is tangent to Σ because it is vertical), ∇_Σ and B are the Riemannian gradient and the second fundamental form of Σ respectively. It follows that if Σ is stable in the sub-Riemannian sense, it is also stable in the Riemannian sense.

The three-dimensional Riemannian Heisenberg group (also known as the three-dimensional Thurston geometry *Nil*) is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its nilpotent Lie group structure:

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\ X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\ X_3 = \frac{\partial}{\partial z}.$$

Theorem 1. *Let a sub-Riemannian structure on Nil be defined by a left-invariant two-dimensional horizontal distribution. Then its normal field should be of the form* $X = \frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda X_1 + \mu X_2 + X_3)$.

If $\lambda = \mu = 0$ *then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if it is a vertical Euclidean plane. In the other case it is minimal if and only if it is a vertical Euclidean plane over a straight line in the* (x, y)*-plane that has the direction* (λ, μ).

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

The three-dimensional Thurston geometry *Sol* is the space $\mathbb{R}^3$ with coordinates (x, y, z) and with the following orthonormal basis of left-invariant vector fields defined by its solvable Lie group structure:

$$X_1 = e^{-z}\frac{\partial}{\partial x},\ X_2 = e^{z}\frac{\partial}{\partial y},\ X_3 = \frac{\partial}{\partial z}.$$

Theorem 2. *Let a sub-Riemannian structure on Sol be defined by a left-invariant two-dimensional horizontal distribution. Then its normal field should be of the form* $X = \frac{1}{\sqrt{\lambda^2+\mu^2+\nu^2}}(\lambda X_1 + \mu X_2 + \nu X_3)$, *where* $\lambda\mu \neq 0$.

If $\nu \neq 0$ *then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if it is cylindrical and can be parameterized either as*

$$r(s,t) = \left(x_0 - \frac{\lambda}{\nu}\, e^{-s}, t, s\right) \quad \text{or as} \quad r(s,t) = \left(t, y_0 + \frac{\mu}{\nu}\, e^{s}, s\right).$$

If $\nu = 0$ *then a complete connected vertical surface is minimal if and only if it is a horizontal Euclidean plane* $z = z_0$ *or* $\lambda = \pm\mu$ *and the surface is a "hyperbolic helicoid" (previously described in* [2]*) with the parameterization*

$$r(s,t) = \left(x_0 + \frac{1}{\sqrt{2}}\, e^{-t} s, y_0 \pm \frac{1}{\sqrt{2}}\, e^{t} s, t\right).$$

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

The three-dimensional Thurston geometry $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the universal covering of the special linear group $\mathrm{SL}(2,\mathbb{R})$. It also can be described as the universal covering of the unit tangent bundle of the hyperbolic plane $\mathbb{H}^2$ with the Sasaki metric. Thus, using the half-plane model of $\mathbb{H}^2$, we can present $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ as the half-space $\{(x,y,z) \in \mathbb{R}^3 \mid y > 0\}$ with the orthonormal frame

$$Y_1 = y\frac{\partial}{\partial x} - \frac{\partial}{\partial z},\ Y_2 = y\frac{\partial}{\partial x},\ Y_3 = \frac{\partial}{\partial z}.$$

Note that the fields Y_1 and Y_2 here are not left-invariant.

Theorem 3. *A two-dimensional horizontal distribution* $\mathcal{H} = X^{\perp}$, *whose normal field* X *is a linear combination of the fields* Y_1*–*Y_3 *with constant coefficients, defines a sub-Riemannian structure on* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *(i.e., is its horizontal distribution) if and only if* X *is of the form* $\frac{1}{\sqrt{\lambda^2+\mu^2+1}}(\lambda Y_1 + \mu Y_2 + Y_3)$, *where* $\lambda \neq -1$. *This sub-Riemannian structure allows vertical minimal surfaces only for* $\lambda = 0$ *and* $\lambda = 1$.

If $\mu \neq 0$ *then a complete connected vertical surface is minimal if and only if it is a half-plane* $x = x_0$ *for* $\lambda = 0$ *or a half-plane* $z = z_0$ *for* $\lambda = 1$.

If $\mu = 0$ *and* $\lambda = 1$ *then a complete connected vertical surface is minimal if and only if it is either a half-plane* $z = z_0$ *or can be parameterized as*

$$r(s,t) = \left(y_0 s \cos t, y_0 \cos t, \sqrt{2}t + z_0\right).$$

If $\mu = \lambda = 0$ *then a complete connected vertical surface in this sub-Riemannian manifold is minimal if and only if is a cylinder over a geodesic in* $\mathbb{H}^2$ *(see, e.g.,* [3]*).*

All these surfaces are stable in the sub-Riemannian sense and thus in the Riemannian sense.

We also find vertical minimal surfaces of a left-invariant sub-Riemannian structure defined by a horizontal distribution $\mathcal{H} = X^{\perp}$, where $X = y\cos z\frac{\partial}{\partial x} + y\sin z\frac{\partial}{\partial y} - \cos z\frac{\partial}{\partial z}$, and establish their stability.

References

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[3] R. Younes Minimal surfaces in $\widetilde{PSL_2}(\mathbb{R})$. *Ill. J. Math.*, 54(2) : 671–712, 2010.