Rational factorization of Lax type flows in the space dual to the centrally extended Lie algebra of fractional integro-differential operators

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The Lie-algebraic approach to the rational factorization of Lax type flows in spaces dual to certain operator Lie algebras (see, for example, [1]) and central extensions of some of them is developed for the central extension of the Lie algebra $\mathbb{A}_\alpha := \mathbb{A}_0\{\{D^\alpha, D^{-\alpha}\}\}$, consisting of fractional integro-differential operators such as $\mathrm{a}_\alpha := \sum_{j\in\mathbb{Z}_+} a_j D^{\alpha(p_\alpha - j)}$, where $\mathbb{A}_0 := A\{\{D, D^{-1}\}\}$ is the Lie algebra of ordinary integral-differential operators, $A := W_2^\infty(\mathbb{R};\mathbb{C}) \cap W_\infty^\infty(\mathbb{R};\mathbb{C})$, $D^\alpha : A \to A$ is a Riemann-Liouville fractional derivative, $\alpha \in \mathbb{C}\setminus\mathbb{Z}$, $\mathrm{Re}\,\alpha \neq 0$, $a_j \in \mathbb{A}_0$, $j \in \mathbb{Z}_+$, $p_\alpha \in \mathbb{N}$ is an order of the fractional integro-differential operator a_α. This Lie algebra possesses the standard commutator $[\mathrm{a}_\alpha, \mathrm{b}_\alpha] = \mathrm{a}_\alpha \circ \mathrm{b}_\alpha - \mathrm{b}_\alpha \circ \mathrm{a}_\alpha$, and invariant with respect to this commutator scalar product $(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \int_{\mathbb{R}} \mathrm{res}_D\,(\mathrm{res}_{D_\alpha}\,(\mathrm{a}_\alpha \circ \mathrm{b}_\alpha D^{-\alpha}))dx$, where $\mathrm{a}_\alpha, \mathrm{b}_\alpha \in \mathbb{A}_\alpha$, "$\circ$" is a symbol of the operator product, res_{D_α} denotes a coefficient at $D^{-\alpha}$ for any fractional integral-differential operator as well as res_D denotes a coefficient at D^{-1} for any ordinary integral-differential operator.

On the central extension $\hat{\mathbb{A}}_\alpha := \bar{\mathbb{A}}_\alpha \oplus \mathbb{C}$ of the parameterized Lie algebra $\bar{\mathbb{A}}_\alpha := \prod_{y\in\mathbb{S}^1} \mathbb{A}_\alpha$ by the Maurer-Cartan 2-cocycle $\omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha) := \langle \mathrm{a}_\alpha, \partial \mathrm{b}_\alpha/\partial y\rangle$, where $\langle \mathrm{a}_\alpha, \mathrm{b}_\alpha\rangle = \int_{\mathbb{S}^1} (\mathrm{a}_\alpha, \mathrm{b}_\alpha)dy$, $\mathrm{a}_\alpha, \mathrm{b}_\alpha \in \bar{\mathbb{A}}_\alpha$, there exist the commutator

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)] = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha], \omega_2(\mathrm{a}_\alpha, \mathrm{b}_\alpha)), \qquad (\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e) \in \hat{\mathbb{A}}_\alpha,$$

and corresponding invariant scalar product

$$((\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)) = \langle \mathrm{a}_\alpha, \mathrm{b}_\alpha\rangle + ed. \tag{1}$$

The Lie algebra $\bar{\mathbb{A}}_\alpha$ allows the splitting into the direct sum of its two Lie subalgebras $\bar{\mathbb{A}}_\alpha = \bar{\mathbb{A}}_{\alpha,+} \oplus \bar{\mathbb{A}}_{\alpha,-}$, where $\bar{\mathbb{A}}_{\alpha,+}$ is the Lie subalgebra of the formal power series by the operator D^α. On the space $\hat{\mathbb{A}}_\alpha^*$ dual to the central extension $\hat{\mathbb{A}}_\alpha$ with respect to the scalar product (1) the $\mathcal{R}$-deformed commutator

$$[(\mathrm{a}_\alpha, d), (\mathrm{b}_\alpha, e)]_{\mathcal{R}} = ([\mathrm{a}_\alpha, \mathrm{b}_\alpha]_{\mathcal{R}}, \omega_{2,\mathcal{R}}(\mathrm{a}_\alpha, \mathrm{b}_\alpha)),$$
$$[\mathrm{a}_\alpha, \mathrm{b}_\alpha]_{\mathcal{R}} = [\mathcal{R}\mathrm{a}_\alpha, \mathrm{b}_\alpha] + [\mathrm{a}_\alpha, \mathcal{R}\mathrm{b}_\alpha], \qquad \omega_{2,\mathcal{R}}(\mathrm{a}_\alpha, \mathrm{b}_\alpha) = \omega_2(\mathcal{R}\mathrm{a}_\alpha, \mathrm{b}_\alpha) + \omega_2(\mathrm{a}_\alpha, \mathcal{R}\mathrm{b}_\alpha),$$

where $\mathcal{R} : \bar{\mathbb{A}}_\alpha \to \bar{\mathbb{A}}_\alpha$ is a space endomorphism, $\mathcal{R} = (P_+ - P_-)/2$, $P_\pm$ are projectors on $\mathbb{A}_{\alpha,\pm}$ accordingly, generates the Lie-Poisson bracket

$$\{\gamma, \mu\}_{\mathcal{R}}(l_\alpha, c) = \langle l_\alpha, [\nabla\gamma(l_\alpha), \nabla\mu(l_\alpha)]_{\mathcal{R}}\rangle + c\omega_{2,\mathcal{R}}(\nabla\gamma(l_\alpha), \nabla\mu(l_\alpha)) := \langle \nabla\gamma(l_\alpha), \Theta\nabla\mu(l_\alpha)\rangle, \tag{2}$$

where $\gamma, \mu \in \mathcal{D}(\bar{\mathbb{A}}_\alpha^*)$ are smooth by Frechet functionals on $\bar{\mathbb{A}}_\alpha^* \simeq \bar{\mathbb{A}}_\alpha$, "$\nabla$" is a symbol of the functional gradient, the Poisson operator $\Theta : T^*(\bar{\mathbb{A}}_\alpha^*) \to T(\bar{\mathbb{A}}_\alpha^*)$ acts by the rule

$$\Theta : \nabla\gamma(l_\alpha) \mapsto -[l_\alpha - c\partial/\partial y, (\nabla\gamma(l_\alpha))_-] + [l_\alpha - c\partial/\partial y, \nabla\gamma(l_\alpha)]_{\leq 0}$$

for any smooth by Frechet functional $\gamma \in \mathcal{D}(\bar{\mathbb{A}}_\alpha^*)$, $T(\bar{\mathbb{A}}_\alpha^*)$ and $T^*(\bar{\mathbb{A}}_\alpha^*)$ are tangent and cotangent spaces to $\bar{\mathbb{A}}_\alpha^*$ accordingly, $(l_\alpha, c) \in \hat{\mathbb{A}}_\alpha^* \simeq \hat{\mathbb{A}}_\alpha$, $l_\alpha \in \bar{\mathbb{A}}_\alpha^*$ is a fractional integro-differential operator of the order $q_\alpha \in \mathbb{N}$. By means of the Casimir invariants $\gamma_n \in I(\hat{\mathbb{A}}_\alpha^*)$, $n \in \mathbb{N}$, satisfying the relationship $[l_\alpha - c\partial/\partial y, \nabla\gamma_n(l_\alpha)] = 0$ at a point $(l_\alpha, c) \in \hat{\mathbb{A}}_\alpha^*$, as Hamiltonians, in the dual space $\hat{\mathbb{A}}_\alpha^* \simeq \hat{\mathbb{A}}_\alpha$ the Lie-Poisson bracket (2) determimes the hierarchy of Lax type Hamiltonian flows in the form

$$\partial l_\alpha/\partial t_n = [(\nabla\gamma_n(l_\alpha))_+, l_\alpha - c\partial/\partial y], \qquad t_n \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{3}$$

where the subscript "+" denotes the projection of the corresponding element from $\bar{\mathbb{A}}_\alpha$ on the Lie subalgebra $\bar{\mathbb{A}}_{\alpha,+}$ and $\nabla\gamma_n(l_\alpha) := \sum_{j\in\mathbb{Z}_+} a_{n,j}D^{\alpha(n-j)}$. One considers another hierarchy of Lax type Hamiltonian flows on the dual space $\hat{\mathbb{A}}^*_\alpha$ such as

$$dl_\alpha/dt_n = [(\nabla\gamma_n(l_\alpha))_+, \tilde{l}_\alpha - c\partial/\partial y], \quad t_n \in \mathbb{R}, \ n \in \mathbb{N}, \tag{4}$$

for some fractional integro-differential operator $\tilde{l} \in \bar{\mathbb{A}}^*_\alpha$ of the order $q_\alpha \in \mathbb{N}$, which is related with the operator $l_\alpha \in \bar{\mathbb{A}}^*_\alpha$ by the generalized gauge transformation

$$\tilde{l}_\alpha(0) - c\partial/\partial y = B_\alpha(0)^{-1}(l_\alpha(0) - c\partial/\partial y)B_\alpha(0), \tag{5}$$

where $B_\alpha(0) \in \bar{\mathbb{A}}_{\alpha,+}$ is some fractional differential operator of the order $s_\alpha \in \mathbb{N}$ with constant coefficients, at the initial moment of the time $t_n \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Theorem 1. *If for every $n \in \mathbb{N}$ at the initial moment of the time $t_n \in \mathbb{R}$ the fractional integro-differential operators $l_\alpha, \tilde{l}_\alpha \in \tilde{\mathrm{g}}^*$ of the order $q_\alpha \in \mathbb{N}$ are related by the relationship* (5)*, there exist such fractional differential operators $A_\alpha, B_\alpha \in \bar{\mathbb{A}}_{\alpha,+}$ of the orders $q_\alpha + s_\alpha$ and s_α accordingly, where $s_\alpha \in \mathbb{Z}_+$, $s_\alpha < q_\alpha$, that the equalities*

$$l_\alpha = A_\alpha B_\alpha^{-1}, \quad \tilde{l}_\alpha = B_\alpha^{-1}(A_\alpha - c\partial B_\alpha/\partial y) \tag{6}$$

hold. The operators $A_\alpha, B_\alpha \in \bar{\mathbb{A}}_{\alpha,+}$ satisfy the following systems of evolution equations

$$\begin{aligned} dA_\alpha/dt_n &= (\nabla\gamma_n(l_\alpha))_+A_\alpha - A_\alpha(\nabla\gamma_n(\tilde{l}))_+ + c(\partial(\nabla\gamma_n(l_\alpha))_+/\partial y)B_\alpha, \\ dB_\alpha/dt_n &= (\nabla\gamma_n(l_\alpha))_+B_\alpha - B_\alpha(\nabla\gamma_n(\tilde{l}_\alpha))_+, \quad n \in \mathbb{N}, \end{aligned} \tag{7}$$

or, equivalently,

$$\begin{aligned} dA_\alpha/dt_n &= (A_\alpha(\nabla\gamma_n(l_\alpha))_-)_+ - ((\nabla\gamma_n(\tilde{l}_\alpha))_-A_\alpha)_+ - c((\partial(\nabla\gamma_n(l_\alpha))_-/\partial y)B_\alpha)_+, \\ dB_\alpha/dt_n &= (B_\alpha(\nabla\gamma_n(l_\alpha))_-)_+ - ((\nabla\gamma_n(\tilde{l}_\alpha))_-B_\alpha)_+, \qquad n \in \mathbb{N}. \end{aligned}$$

which possess an infinite sequence of the conservation laws $H_n \in \mathcal{D}(\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+})$, $n \in \mathbb{N}$, in the forms

$$H_n(A_\alpha, B_\alpha) := \gamma_n(l_\alpha)|_{l_\alpha = A_\alpha B_\alpha^{-1}} = \gamma_n(\tilde{l}_\alpha)|_{\tilde{l}_\alpha = B_\alpha^{-1}(A_\alpha - c\partial B_\alpha/\partial y)}.$$

Theorem 2. *For every $n \in \mathbb{N}$ the system of evolution equations* (7)*, given on the subspace $\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+} \subset \bar{\mathbb{A}}_\alpha \times \bar{\mathbb{A}}_\alpha$, is Hamiltonian with respect to the Poisson bracket $\{.,.\}_{\mathcal{L}}$ which arises as a reduction of the Poisson bracket $\{.,.\}_{\bar{\mathcal{L}}}$ with the corresponding Poisson operator $\bar{\mathcal{L}} = (P')^{-1}(\Theta \oplus \tilde{\Theta})(P'^*)^{-1}$, where $\tilde{\Theta}$ is a Poisson operator generating the Poisson bracket* (2) *at a point $\tilde{l} \in \bar{\mathbb{A}}^*_\alpha$, $P'^* : T^*(\bar{\mathbb{A}}^*_\alpha \oplus \bar{\mathbb{A}}^*_\alpha) \to T^*(\bar{\mathbb{A}}_\alpha \times \bar{\mathbb{A}}_\alpha)$ is an operator adjoint to the Frechet derivative $P' : T(\bar{\mathbb{A}}_\alpha \times \bar{\mathbb{A}}_\alpha) \to T(\bar{\mathbb{A}}^*_\alpha \oplus \bar{\mathbb{A}}^*_\alpha)$ of the Backlund transformation $P : (A_\alpha, B_\alpha) \in \bar{\mathbb{A}}_\alpha \times \bar{\mathbb{A}}_\alpha \mapsto (l_\alpha, \tilde{l}_\alpha) \in \bar{\mathbb{A}}^*_\alpha \oplus \bar{\mathbb{A}}^*_\alpha$, determined by the equalities* (6)*, $(P'^*)^{-1}$ is inverse to one, on $\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+}$ and Hamiltonians $\bar{H}_n \in \mathcal{D}(\bar{\mathbb{A}}_{\alpha,+} \times \bar{\mathbb{A}}_{\alpha,+})$ in the forms*

$$\bar{H}_n(A_\alpha, B_\alpha) := \gamma_n(l_\alpha)|_{l_\alpha = A_\alpha B_\alpha^{-1}} + \gamma_n(\tilde{l}_\alpha)|_{\tilde{l}_\alpha = B_\alpha^{-1}(A_\alpha - c\partial B_\alpha/\partial y)}, \qquad n \in \mathbb{N}.$$

In the case of $c = 0$ the second Hamiltonian representation for the hierarchy (7) is also found.

The rational factorization method for the central extension $\hat{\mathbb{A}}_\alpha$ is applied to construct a new integrable hierarchy of two-dimensional nonlinear dynamical systems with fractional derivatives by one spatial variable as well as a new integrable hierarchy of two-dimensional hydrodynamic Benney-type systems, which is its quasiclassical approximation.

References

[1] Oksana Hentosh, Anatolij Prykarpatski. Rational factorization of Hamiltonian flows in the space dual to the Lie algebra of fractional integro-differential operators and integrable Benney-type hydrodynamic systems. *J. of Mathematical Sciences*, 279(3) : 308–329, 2024.