Criteria of optimality of some classes of simple functions on surfaces with the boundary

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Smooth functions are the tool of research in many scientific fields. Their classification and optimality problems are important enough. There are a number of papers dedicated to functions with non-degenerated critical points on the boundary of a surface [1, 3, 4, 5, 7, 8] and with inner [2, 10] and boundary [6, 9] isolated critical points on the low-dimensional manifolds.

A function is *optimal* if it has the smallest number of critical points among all functions on present surface (if such exists). Also the function, which has no more than one critical point on each level line, is called a *simple function*.

A simple Morse function being defined on a surface with the boundary, is called a *mm-function*, if its restriction to the boundary is also a Morse function and all critical points belong to the boundary of the surface.

In this thesis we have presented the criterias of optimality of the following classes of simple functions: (1) mm-functions on a surface with the boundary; (2) Morse functions on a closed oriented connected surface; (3) functions with isolated critical points on the boundary of a connected surface with the connected boundary.

Theorem 1. *A mm-function being defined on a surface of genus g with k components of the boundary is optimal if and only if it has $4g + 2k$ critical points if the surface is oriented and $2g + 2k$ critical points if the surface is non-oriented.*

A function is polar if it has exactly one minimum and one maximum point on a present manifold.

Theorem 2. *A Morse function on a closed oriented surface is optimal i and only if it is polar on a present surface.*

Theorem 3. *Mm-function on a smooth compact oriented surface with the boundary is optimal if and only if it is polar on a present surface.*

Let M be a smooth compact oriented surface with the connected boundary ∂M and let $f : M \to \mathbb{R}$ be a smooth function defined on M with finitely many critical points on the boundary. Remark that the number of critical points is finite is equivalent to their isolatedness. Let $CP(f)$ ($ICP(f)$) be a set of (isolated) critical points of the function f and f_∂ be a restriction of the function f to the boundary ∂M of the surface M. Then we are going to consider the following set of functions:

$$\Theta(M) = \{f : M \to \mathbb{R} | f \in C^\infty(M), CP(f) = ICP(f) = ICP(f_\partial), f \text{ is simple}\}$$

Theorem 4. *Suppose that $f \in \Theta(M)$ and M is a connected compact surface with connected boundary, which is not homeomorphic to a two-dimensional disk. Then the function f is optimal if and only if it has exactly three critical points. Then the function f is optimal if and only if it has exactly two critical points in the case of two-dimension disk.*

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