Reconstructing Morse functions with prescribed preimages of single points

Naoki Kitazawa
(Osaka Central Advanced Mathematical Institute)
E-mail: naokikitazawa.formath@gmail.com

Morse functions have been fundamental and strong tools in geometry. Morse functions are also important and interesting objects in geometry. For Morse functions and handles, which are fundamental tools and objects, see [5] for example.

We consider the following fundamental problem. This was essentially started in [1].

Problem 1. Let $m > 1$ be an integer and $a_1 < a_2$ real numbers. Let F_1 and F_2 be smooth closed manifolds of dimension $m-1$. Can we reconstruct a Morse function $\tilde{f}_{a_1,a_2} : \tilde{M}_{a_1,a_2} \to \mathbb{R}$ on some m-dimensional compact and connected manifold $\tilde{M}_{a_1,a_2}$ onto the closed interval $[a_1, a_2]$ enjoying the following?

(1) The boundary of $\tilde{M}_{a_1,a_2}$ is diffeomorphic to $F_1 \sqcup F_2$ and the preimage $\tilde{f}_{a_1,a_2}^{-1}(a_i)$ is diffeomorphic to F_i.
(2) There exists a unique critical value a and a is in the open interval (a_1, a_2). The preimage $\tilde{f}_{a_1,a_2}^{-1}(a)$ is connected.

In the case $m = 2$, F_i must be a disjoint union of circles and related studies had been presented before. For this, Sharko [8] first considered the reconstruction of smooth functions with critical points being represented by some elementary polynomials, on closed surfaces. Later, Michalak [6] has explicitly solved Problem 1 in the case $m = 2$ (and $F_i = \sqcup S^{m-1}$ where S^{m-1} is the $(m-1)$-dimensional unit sphere). There, he has also classified Morse functions on given closed surfaces in terms of their *Reeb graphs*: the *Reeb graph* of a smooth function $c : X \to \mathbb{R}$ on a manifold X with no boundary is the quotient space W_c of the manifold obtained by the equivalence relation regarding that two points are equivalent if and only if they are in a same connected component of a same preimage $c^{-1}(y)$. They are classical objects and have already appeared in [7]. They have been fundamental and strong tools in understanding the manifolds compactly.

We present our study and result.

Definition 2 ([3])**.** A *most fundamental handlebody* of dimension m is a smooth, compact and connected manifold diffeomorphic to one obtained by attaching finitely many handles to the boundary S^{m-1} of the m-dimensional unit disk D^m disjointly and simultaneously where at least one handle is attached.

Example 3. The unit disk D^m and a compact manifold represented as a boundary connected sum $\natural_j(S^{k_j} \times D^{m-k_j})$ $(1 \leq k_j \leq m-1)$ (considered in the smooth category) are m-dimensional most fundamental handlebodies.

Theorem 4 ([1, 2])**.** *In the case each of connected components of F_1 and F_2 is the boundary of some most fundamental handlebody of dimension m, Problem 1 is affirmatively solved.*

A main ingredient of the proof is as follows: by attaching handles to $F_1 \times \{1\} \subset F_1 \times [0, 1]$ disjointly, simultaneously and suitably, we have an m-dimensional smooth compact and connected manifold $\tilde{M}_{a_1,a_2}$ whose boundary is diffeomorphic to $F_1 \sqcup F_2$. Note that [1] also shows local functions around local extrema which belong to a certain class generalizing the class of Morse(-Bott) functions, as another result.

Corollary 5 ([3]). *In the case $m = 4$ with F_j $(j = 1, 2)$ being orientable, Problem 1 is affirmatively solved.*

This comes from fundamental and important facts on 3-dimensional manifold theory.

We review elementary properties of closed surfaces and introduce some elementary numerical invariants. A closed and connected surface F is diffeomorphic to a connected sum of the form $(\sharp_{j_1=1}^{k_1}(S^1 \times S^1))\sharp(\sharp_{j_2=1}^{k_2}(\mathbb{R}P^2))$ where k_1 and k_2 are non-negative integers. A closed and connected surface F is orientable if and only if $k_2 = 0$. We can define $P(F) = k_2$ as a topological invariant for closed and connected surfaces and we can extend this to closed surfaces F which may not be connected in the additive way. Note that if $P(F)$ is odd, then this is not the boundary of any 3-dimensional compact manifold. We can define another topological invariant $P_o(F)$ for closed surfaces F as the numbers of connected components F_j of F with $P(F_j)$ being odd.

Theorem 6 ([2, 4]). *Problem 1 is solved affirmatively in the case $m = 3$ if and only if either the following three hold.*

(1) $P_o(F_1) = P_o(F_2)$.
(2) $P_o(F_1) - P_o(F_2)$ *is even,* $P_o(F_1) > P_o(F_2)$, *and* $P_o(F_1) \leq P(F_2)$.
(3) $P_o(F_1) - P_o(F_2)$ *is even,* $P_o(F_1) < P_o(F_2)$, *and* $P(F_1) \geq P_o(F_2)$.

For this, the case F_i are orientable is a specific case of Theorem 4. The condition has been shown to be sufficient in [2] first by explicit construction of Morse functions. [4] has shown that the condition is also a necessary condition by investigating attachment of handles to $F_1 \times \{1\} \subset F_1 \times [0, 1]$ to have a smooth, compact and connected manifold $\tilde{M}_{a_1,a_2}$ whose boundary is diffeomorphic to $F_1 \sqcup F_2$, precisely. [4] is also a kind of addenda to [2].

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