Umbilics on complete convex planes : The Toponogov Conjecture

Wilhelm Klingenberg
(Dept Mathematics, Durham University, DH13LE, UK)
E-mail: dma0wk@gmail.com

In 1995, Victor Andreevich Toponogov [1] authored the following conjecture: "Every smooth non-compact strictly convex and complete surface of genus zero has an umbilic point, possibly at infinity". In our talk, we will outline the 2024 proof in collaboration with Brendan Guilfoyle [2].

Theorem 1. [2] *Assume that $P \hookrightarrow \mathbb{R}^3$ be a proper embedded strictly convex surface, and assume that it is diffeomorphic to the plane and $C^{3,\alpha}$ - regular. Then*

$$\inf_{p\in P} |\kappa_1(p) - \kappa_2(p)| = 0.$$

Namely we prove that, should a counter-example to the Conjecture exist, (a) the Fredholm index of an associated Riemann Hilbert boundary problem for holomorphic discs is negative [3]. Thereby, (b) no such holomorphic discs survive for a generic perturbation of the boundary condition (these form a Banach manifold under the assumption that the Conjecture is incorrect). Finally, however, (c) the geometrization by a neutral Kaehler metric [4] of the associated model allows for Mean Curvature Flow [5] with mixed Dirichlet – Neumann boundary conditions to generate a holomorphic disc from an initial spacelike disc. This completes the indirect proof of said conjecture as (b) and (c) are in contradiction. Note that our regularity assumption is stronger than the minimal required in the context, which would be twice continuously differentiable.

References

[1] V. A. Toponogov *On conditions for existence of umbilical points on a convex surface* , Siberian Mathematical Journal, vol 36 p 780–784, 1995

[2] B. Guilfoyle, W. Klingenberg, *Proof of the Toponogov Conjecture on complete surfaces*, Gokova Geom. Topol. GGT, 17 (2024) 1–50.

[3] B. Guilfoyle, W. Klingenberg, *Fredholm-regularity of holomorphic discs in plane bundles over compact surfaces.*, Annales de la Faculté des sciences de Toulouse (En ligne), (2020) 29(3), 565-576.

[4] B. Guilfoyle, W. Klingenberg, *An indefinite Kaehler metric on the space of oriented lines*, J. London Math. Soc. (2005) 72.2, 497–509.

[5] B. Guilfoyle and W. Klingenberg, *Higher codimensional mean curvature flow of compact spacelike submanifolds*, Trans. Amer. Math. Soc. 372.9 (2019) 6263–6281.