The p-adic class numbers of Z-covers of graphs

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In the spirit of arithmetic topology, we propose to study the p-adic limit values of the number of spanning trees in pro-p covers of graphs. This talk is based on a joint work [KU25] and will focus on a specific example.

Let X be a *finite connected graph*, that is, a 1-dimensional CW complex. A *spanning tree* T of X is a connected subgraph that contains all vertices and no loops. The number of spanning trees of each X is denoted by $k(X)$. A basic reference for graphs is [Ter11].

Suppose that X is *the 8-graph*, consisting of one vertex and two looped edges. Let s_1, s_2 denote the elements of the fundamental group $\pi_1(X)$ represented by the two loops. We consider a specific surjective homomorphism

$$\varphi : \pi_1(X) \to \mathbb{Z};\ s_1 \mapsto 1, s_2 \mapsto 2.$$

The $\mathbb{Z}$-cover $X_\infty \to X$ corrreponding to $\operatorname{Ker}\varphi$ is so-called *the Fibonacci tower*. The adjacency matrix yields *the Ihara zeta function* and *the Ihara polynomial* $I(t) = 4 - (t + 1/t) - (t^2 + 1/t^2)$. We further put $J(t) := t^2 I(t)/(t-1)^2 = t^2 + 3t + 1$.

For each $n \in \mathbb{Z}_{>0}$, let $X_n \to X$ denote the $\mathbb{Z}/n\mathbb{Z}$-subcover. Then, Pengo–Vallieres [PV25, Theorem 3.6] asserts that the number of spanning trees of X_n may be calculated by using the cyclic resultant $\operatorname{Res}(t^n - 1, J(t)) = \prod_{\zeta^n=1} J(\zeta) \in \mathbb{Z}$ as

$$k(X_n) = k(X) n^{2-1} |\operatorname{Res}(t^n - 1, J(t))| / J(1).$$

On the other hand, p being a prime number, Kisilevsky [Kis97] and Ueki–Yoshizaki [UY25] proved that p-power-th cyclic resultant p-adically converges in the ring of p-adic integers $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ and gave explicit formulae. For instanse, if $\Phi_m(t) \in \mathbb{Z}[t]$ denote the m-th cyclotomic polynomial and $f(t) \in \mathbb{Z}[t]$ satisfies $f(t) \equiv \Phi_m(t) \bmod p$, then

$$\lim_{n\to\infty} \operatorname{Res}(t^{p^n} - 1, f(t)) = \Phi_m(1)$$

holds.

Combining the above, we obtain the following for the 8-graph X.

Theorem 1. *The sequence $(k(X_{p^n}))_n$ converges in $\mathbb{Z}_p$. We have*

$$\lim_{n\to\infty} k(X_{p^n})/p^n \in \mathbb{Q} \underset{\text{iff}}{\iff} p = 2, 3, 5.$$

In addition, if we put $r_n := |\operatorname{Res}(t^n - 1, J(t))|$, then we have

$$\lim_{n\to\infty} |r_{2^n}| = -3,\ \lim_{n\to\infty} |r_{3^n}| = 2,\ \lim_{n\to\infty} |r_{5^n}| = 0.$$

The non-5 part of $|r_{5^n}|$ is $|r_{5^n}|/5^{2n+1}$. Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \widehat{\overline{\mathbb{Q}_5}}$ of an algebraic closure of $\mathbb{Q}$ into the completion of an algebraic closure of the 5-adic number field. Let α, β denote the roots of $J(t)$

and let log denote the 5-adic logarithm. Then we have

$$\lim_{n\to\infty} |r_{5^n}|/5^{2n+1} = \frac{\log\alpha\log\beta}{5} \in \mathbb{Z}_5.$$

We may observe that these sequences converge quickly:

If $p = 2$,

n	1	2	3	4	5	6
$-\mathrm{Res}(t^{2^n}-1, J(t))$	5	45	2205	4870845	23725150497405	...
$-\mathrm{Res}(t^{2^n}-1, J(t)) \bmod 2^n$	-3	-3	-3	-3	-3	-3

.

If $p = 3$,

n	1	2	3	4	5	6
$\mathrm{Res}(t^{3^n}-1, J(t))$	20	5780	192900153620	...	...	...
$\mathrm{Res}(t^{3^n}-1, J(t)) \bmod 3^n$	2	2	2	2	2	2

.

If $p = 5$,

n	1	2	3	4	5	6
$\mathrm{Res}(t^{5^n}-1, J(t))$	5^3	$5^5\cdot 3001^2$	$5^7\cdot 3001^2\cdot 158414167964045700001^2$	...	...	...
$\frac{1}{5^{2n+1}}\mathrm{Res}(t^{5^n}-1, J(t)) \bmod 5^n$	1	1	1	376	2876	15376

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