Weyl algebras and generalized symmetries

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Weyl algebras are fundamental objects in ring theory that arise from various perspectives in mathematics and physics, and the development of their theory is related to such names as Dirac, Heisenberg, Littlewood, Weyl, Segal, Dixmier and Kashiwara. Their feature is capturing the noncommutativity of differential operators with polynomial coefficients, which makes these algebras ubiquitous in abstract algebra, noncommutative geometry, representation theory and quantum mechanics. The representation theory of Weyl algebras led to the development of the so-called algebraic analysis, an advanced branch of algebra within whose framework several long-standing conjectures have been proven.

Following [1, 3], let $\mathbb{K}$ be a field of characteristic zero. The first Weyl algebra A_1 is the associative algebra over $\mathbb{K}$ generated by elements x and ∂ that satisfy the defining relation $\partial x - x\partial = 1$. The Weyl algebra A_1 is a central, simple, Noetherian, hereditary domain of Gelfand–Kirillov dimension two which is canonically isomorphic to the ring of differential operators $\mathbb{K}[x][\frac{\mathrm{d}}{\mathrm{d}x}]$ with coefficients from the polynomial ring $\mathbb{K}[x]$. The Bergman's diamond lemma [2] allows one to easily show that the tuple $(x^k\partial^l \mid k,l\in\mathbb{N}_0)$ is a basis of A_1. The nth Weyl algebra A_n is the tensor product $A_1\otimes\cdots\otimes A_1$ of n copies of the first Weyl algebra.

From the perspective of symmetry analysis of differential equations, the first and the second real Weyl algebras arise as the algebras of linear generalized symmetries of the linear (1+1)-dimensional heat equation $u_t=u_{xx}$ and of the remarkable (1+2)-dimensional Fokker–Planck equation $u_t+xu_y=u_{xx}$, see [5] and [8], respectively. The above is only one way the close relationship between these two equations manifests itself. This relationship was revealed in the course of extended symmetry analysis of the latter and former equations in [5, 6, 7] and [4, 8], but it can in fact be embedded in a broader framework.

For each $n\in\mathbb{N}$, consider the class $\mathcal{U}_n$ of (ultra)parabolic linear second-order partial differential equations with $1+n$ independent variables t, x_1, ..., x_n and dependent variable u, where the corresponding (symmetric) matrices of coefficients of second-order derivatives of the dependent variable u are of rank one, and the number $n+1$ of independent variables is essential: none among them plays the role of a parameter even up to their point transformations. The equation

$$\mathcal{F}_n\colon\quad u_t+x_1u_{x_2}+\cdots+x_{n-1}u_{x_n}=u_{x_1x_1}$$

belongs to the class $\mathcal{U}_n$. Notably, the equations $\mathcal{F}_1$ and $\mathcal{F}_2$ coincide with the above linear heat and remarkable Fokker–Planck equations, respectively. The classes $\mathcal{U}_1$ and $\mathcal{U}_2$ coincide with the classes of parabolic linear second-order partial differential equation with two independent variables and of ultraparabolic linear second-order partial differential equations with three independent variables, respectively.

In this talk, we present the results of our in-depth preliminary analysis of the properties of the equations $\mathcal{F}_n$ within their respective classes $\mathcal{U}_n$. Among many surprising observations and conjectures, there are the following:

- The dimension of the essential Lie invariance algebra $\mathfrak{g}_n^{\rm ess}$ of $\mathcal{F}_n$ is equal to $2n+4$, and this algebra is isomorphic to the algebra ${\rm sl}(2,\mathbb{R})\mathbin{\in}_{\rho_{2n-1}\oplus\rho_0}{\rm h}(n,\mathbb{R})$. The Levi factor $\mathfrak{f}_n$ and the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ are isomorphic to the real degree-two special linear algebra ${\rm sl}(2,\mathbb{R})$ and the rank-n Heisenberg algebra ${\rm h}(n,\mathbb{R})$, respectively. Here ρ_m denotes the standard real irreducible representation of ${\rm sl}(2,\mathbb{R})$ in the $(m+1)$-dimensional vector space.
- The dimension of $\mathfrak{g}_n^{\rm ess}$ is maximal among those of the essential Lie invariance algebras of equations from the class $\mathcal{U}_n$, and each equation whose essential Lie invariance algebra is of this maximal dimension is reduced to $\mathcal{F}_n$ by a point transformation in the space $\mathbb{R}^{1+n}_{t,x_1,\dots,x_n}\times\mathbb{R}_u$.
- The essential point-symmetry group $G_n^{\rm ess}$ of the equation $\mathcal{F}_n$ is isomorphic to the Lie group $\big({\rm SL}(2,\mathbb{R})\ltimes_{\varrho_{2n-1}\oplus\varrho_0}{\rm H}(n,\mathbb{R})\big)\times\mathbb{Z}_2$, where ${\rm H}(n,\mathbb{R})$ denotes the rank-n Heisenberg group and ϱ_m is the irreducible representation of the real degree-two special linear group ${\rm SL}(2,\mathbb{R})$ in $\mathbb{R}^{m+1}$.
- A complete list of discrete point symmetry transformations of the equation $\mathcal{F}_n$ that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution I alternating the sign of u, $I\colon(t,x_1,\dots,x_n,u)\mapsto(t,x_1,\dots,x_n,-u)$. Thus, the quotient group of the complete point-symmetry pseudogroup G_n of $\mathcal{F}_n$ with respect to its identity component is isomorphic to $\mathbb{Z}_2$.
- The algebra of canonical representatives of generalized symmetries of $\mathcal{F}_n$ is $\Sigma_n=\Lambda_n\mathbin{\in}\Sigma_n^{-\infty}$. Here Λ_n is the subalgebra of linear generalized symmetries of $\mathcal{F}_n$, which is generated by acting with the Lie-symmetry operators associated with the canonical basis of the complement of the center $\langle u\partial_u\rangle$ in the (nil)radical $\mathfrak{r}_n$ of $\mathfrak{g}_n^{\rm ess}$ on the elementary seed symmetry vector field $u\partial_u$, and $\Sigma_n^{-\infty}$ is the ideal associated with linear superposition of solutions of $\mathcal{F}_n$.
- The algebra Λ_n is isomorphic to the Lie algebra $A_n^{(-)}$ associated with the nth Weyl algebra A_n.
- A generalized vector field is a master symmetry of $\mathcal{F}_n$ in the sense of the definition given in [7, p. 315] if and only if up to a triviality equivalence relation, it is a generalized symmetry of $\mathcal{F}_n$.
- The algebra Λ_n is two-generated as a Lie algebra, i.e., there is a pair of its elements such that Λ_n coincides with its subalgebra containing all successive commutators (aka nonassociative monomials) of these two elements.

This work introduces a substantial research program aimed at a deeper understanding of the symmetry properties of linear second-order partial differential equations.

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