On the prime ends extension of unclosed inverse mappings

Victoria Desyatka
(Zhytomyr Ivan Franko State University)
E-mail: victoriazehrer@gmail.com

Zarina Kovba
(Zhytomyr Ivan Franko State University)
E-mail: victoriazehrer@gmail.com

Evgeny Sevost'yanov
(Zhytomyr Ivan Franko State University; Institute of Applied Mathematics and Mechanics, Slov'yans'k)
E-mail: esevostyanov2009@gmail.com

The following statements contain itself some results on prime end boundary extension of quasiconformal mappings.

Theorem A. *Under a quasiconformal mapping f of a collared domain D_0 onto a domain D, there exists a one-to-one correspondence between the boundary points of D_0 and the prime ends of D. Moreover, the cluster set $C(f,b)$, $b\in\partial D_0$, coincides with the impression $I(P)$ of the corresponding prime end P of D (see* [1, Theorem 4.1]*).*

Given $f:D\rightarrow D^{\,\prime}$, we set $C(f,x):=\{y\in \overline{{\Bbb R}^n}:\exists\,x_k\in D: x_k\rightarrow x, f(x_k)\rightarrow y, k\rightarrow\infty\}$ and $C(f,\partial D)=\bigcup\limits_{x\in \partial D}C(f,x)\,.$

Theorem B. *Let $f:D\rightarrow {\Bbb R}^n$ be quasiregular mapping with $C(f, \partial D)\subset \partial f(D)$. If D is locally connected at a point $b\in \partial D$ and $D^{\,\prime}=f(D)$ is qc accessible at some point $y\in C(f, b)$, then $C(f, b)=\{y\}$ (see* [2, Theorem 4.2]*).*

The goal of this abstract is to consider mappings which are not closed. Let $y_0\in {\Bbb R}^n$, $0<r_1<r_2<\infty$ and $A=A(y_0, r_1,r_2)=\left\{ y\,\in\,{\Bbb R}^n: r_1<|y-y_0|<r_2\right\}$. Given sets E, $F\subset\overline{{\Bbb R}^n}$ and a domain $D\subset {\Bbb R}^n$ we denote by $\Gamma(E,F,D)$ a family of all paths $\gamma:[a,b]\rightarrow \overline{{\Bbb R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in\,F$ and $\gamma(t)\in D$ for $t \in (a, b)$. If $f:D\rightarrow {\Bbb R}^n$, $y_0\in f(D)$ and $0<r_1<r_2<d_0=\sup\limits_{y\in f(D)}|y-y_0|$, then by $\Gamma_f(y_0, r_1, r_2)$ we denote the family of all paths γ in D such that $f(\gamma)\in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0,r_1,r_2))$. Let $Q:{\Bbb R}^n\rightarrow [0, \infty]$ be a Lebesgue measurable function. We say that f *satisfies Poletsky inverse inequality* at the point $y_0\in f(D)$, if the relation

$$M(\Gamma_f(y_0, r_1, r_2))\leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^n(|y-y_0|)\, dm(y) \tag{1}$$

holds for any Lebesgue measurable function $\eta: (r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\, dr\geqslant 1\,. \tag{2}$$

Recall that a mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of each point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *is open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n$. Later, in the extended space $\overline{{{\Bbb R}}^n}={{\Bbb R}}^n\cup\{\infty\}$ we use the spherical (chordal) metric h (see [3, Definition 12.1]). Further, the closure $\overline{A}$ and the boundary ∂A of the set $A\subset\overline{{\Bbb R}^n}$ we understand relative to the chordal metric h in $\overline{{\Bbb R}^n}$.

The boundary of D is called *weakly flat* at the point $x_0\in \partial D$, if for every $P>0$ and for any neighborhood U of the point x_0 there is a neighborhood $V\subset U$ of the same point such that $M(\Gamma(E, F, D))>P$ for any continua $E, F\subset D$ such that $E\cap\partial U\ne\varnothing\ne E\cap\partial V$ and $F\cap\partial U\ne\varnothing\ne F\cap\partial V$. The boundary of D is called weakly flat if the corresponding property holds at any point of the boundary D. Consider the following definition, see e.g. [1]. The boundary of a domain D in ${\Bbb R}^n$ is said to be *locally quasiconformal* if every $x_0\in\partial D$ has a neighborhood U that admits a quasiconformal mapping φ onto the unit ball ${\Bbb B}^n\subset{\Bbb R}^n$ such that $\varphi(\partial D\cap U)$ is the intersection of ${\Bbb B}^n$ and a coordinate hyperplane. The sequence of cuts $\sigma_m,$ $m=1,2,\ldots,$ is called *regular,* if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in {\Bbb N}$ and, in addition, $d(\sigma_{m})\rightarrow 0$ as $m\rightarrow\infty.$ If the end K contains at least one regular chain, then K will be called *regular.* We say that a bounded domain D in ${\Bbb R}^n$ is *regular*, if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in ${\Bbb R}^n,$ and, besides that, every prime end in D is regular. Note that space $\overline{D}_P=D\cup E_D$ is metric, which can be demonstrated as follows. If $g:D_0\rightarrow D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain $D,$ then for $x, y\in \overline{D}_P$ we put:

$$\rho(x, y):=|g^{\,-1}(x)-g^{\,-1}(y)|\,, \tag{3}$$

where the element $g^{\,-1}(x),$ $x\in E_D,$ is to be understood as some (single) boundary point of the domain $D_0.$ The specified boundary point is unique, see e.g. [1, Theorem 4.1]. It is easy to verify that ρ in (3) is a metric on $\overline{D}_P.$

Let $E\subset\overline{D}.$ We say that D is *finitely connected at the point* $z_0\in E,$ if for each neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V}\subset\widetilde{U}$ of z_0 such that $(D\cap\widetilde{V})\setminus E$ consists of finite number of components. We say that D is *finitely connected on* $E,$ if D is finitely connected at every point $z_0\in E.$ The following theorem is true.

Theorem 1. *Let D and $D^{\,\prime}$ be domains in ${\Bbb R}^n,$ $n\geqslant 2,$ and let D be a domain with a weakly flat boundary. Suppose that f is open discrete mapping of D onto $D^{\,\prime}$ satisfying the relation (1) at each point $y_0\in \overline{D^{\,\prime}}.$ In addition, assume that the following conditions are fulfilled:*

1) for each point $y_0\in\partial D^{\,\prime}$ there is $0<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ such that for any $0<r_1<r_2<r_0:=\sup\limits_{y\in D^{\,\prime}}|y-y_0|$ there exists a set $E\subset[r_1, r_2]$ of positive linear Lebesgue measure such that Q is integrable on $S(y_0, r)$ for $r\in E;$

2) $D^{\,\prime}$ is a regular domain and, in addition, $D^{\,\prime}$ is finitely connected on $C(f, \partial D)\cap D^{\,\prime},$ i.e., for each point $z_0\in C(f, \partial D)\cap D^{\,\prime}$ and for any neighborhood U of this point there exists a neighborhood $V\subset U$ of this point such that the set $V\setminus C(f, \partial D)$ consists of a finite number of components;

3) the set $f^{\,-1}(C(f, \partial D)\cap D^{\,\prime})$ is nowhere dense in $D;$

4) the set $D^{\,\prime}$ is finitely connected in $E_{D^{\,\prime}}:=\overline{D^{\,\prime}}_P\setminus D^{\,\prime},$ i.e., for any $P_0\in E_{D^{\,\prime}}$ and any neighborhood U of P_0 in $\overline{D^{\,\prime}}_P$ there is a neighborhood $V\subset U$ such that $V\setminus C(f, \partial D)$ consists of finite number of components.

Then the mapping f has a continuous extension $\overline{f}:\overline{D}\rightarrow\overline{D^{\,\prime}}_P$ by the metric ρ defined in (3). Moreover, $\overline{f}(\overline{D})=\overline{D^{\,\prime}}_P.$

References

[1] Näkki R. Prime ends and quasiconformal mappings. *J. Anal. Math.*, 35: 13–40, 1979.
[2] Srebro U. Conformal capacity and quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I. Math.*, 529: 1–8, 1973.
[3] Väisälä J. *Lectures on n-Dimensional Quasiconformal Mappings.* Lecture Notes in Math. 229. Berlin etc., Springer–Verlag, 1971.