Tree maps with acyclic Markov graphs

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Markov graphs provide an interesting tool in combinatorial dynamics, which helps to establish Sharkovsky-type results for continuous vertex maps on topological trees [1].

From purely discrete point of view, the construction of Markov graphs stems from a given vertex self-map on a combinatorial tree. Namely, let X be a tree and $\sigma : V(X) \to V(X)$ be a map. The corresponding Markov graph is a directed graph having the edge set $E(X)$ as its vertex set, with the arc set $\{(uv, xy) : x, y \in [\sigma(u), \sigma(v)]_X\}$ (here $[a, b]_G = \{x \in V(G) : d_G(a, x) + d_G(x, b) = d_G(a, b)\}$ denotes the metric interval between a, b in a connected graph G).

In other words, the vertices of $\Gamma(X, \sigma)$ are the edges of X with the existence of an arc $uv \to xy$ if only if uv "covers" xy under the map σ.

Here we are interested in maps on trees with acyclic Markov graphs. Note that tree maps with irreflexive Markov graphs are called anti-expansive. It can be proved that each anti-expansive map has a unique fixed point [2]. In case of acyclic Markov graphs, we can say much more.

Theorem 1. *Let X be a tree and $\sigma\colon V(X) \to V(X)$ be its vertex self-map. Then $\Gamma(X, \sigma)$ is acyclic if and only if there exists a "filtration" of subtrees $X = X_0 \supset X_1 \supset \cdots \supset X_m$ such that*

(1) $V(X_m) = \{u_0\}$ is a singleton with u_0 being a fixed point for f;

(2) $\sigma(V(X_k)) \subseteq V(X_{k+1})$ for $0 \le k \le m-1$.

A map $\sigma : V \to V$ is called nilpotent if there is $k \ge 1$ such that σ^k is constant. The next result completely describes the dynamical structure of maps on trees with acyclic Markov graphs.

Proposition 2. *A map $\sigma : V \to V$ is nilpotent if and only if there is a tree X on V such that $\Gamma(X, \sigma)$ is acyclic.*

In [3], the characterization of trees X which admit maps σ with $\Gamma(X, \sigma)$ being a path was obtained (these are the so-called balanced spiders).

A digraph is called an M-graph provided it is isomorphic to some Markov graph for a map on a tree. It can be proved that in-trees, out-trees, orientations of paths and stars are all M-graphs. However, not every polytree is an M-graph. To see this, consider the spider T obtained by gluing three copies of P_3 by their leaf vertices. Let D denotes the bipartite orientation of T in which the center of X becomes a source. Then it can be showed that D is not an M-graph.

Conjecture: Any polytree with out-degrees bounded by 2 is an M-graph.

References

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