Homotopy properties of Morse functions on non-orientable surfaces

Iryna Kuznietsova
(Institute of Mathematics, NAS of Ukraine, Tereshchenkivska str. 3, Kyiv, 01024 Ukraine)
E-mail: kuznietsova@imath.kiev.ua

Sergiy Maksymenko
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: maks@imath.kiev.ua

Let M be a smooth compact surface and P is a real line $\mathbb{R}$ or a circle S^1. Denote by $\mathcal{F}(M, P)$ the space of smooth functions $f \in C^\infty(M, P)$ satisfying the following conditions:

1) the function f takes constant value at ∂M and has no critical point in ∂M;
2) for every critical point z of f there is a local presentation $f_z : \mathbb{R}^2 \to \mathbb{R}$ of f near z such that f_z is a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

Let X be a closed subset of M. Denote by $\mathcal{D}(M, X)$ the group of C^∞-diffeomorphisms of M fixed on X, that acts on the space of smooth functions $C^\infty(M, P)$ by the rule: $(f, h) \longmapsto f \circ h$, where $h \in D(M, X)$, $f \in C^\infty(M, P)$.

Let $\mathcal{O}(f, X) = \{f \circ h \mid h \in D(M, X)$ be *orbit* of f with respect to the action above.

Precise algebraic structure of such orbits for oriented surfaces was described in [1]. In particular, the following theorem holds.

Theorem 1. [1] *Let M be a connected compact oriented surface except 2-sphere and 2-torus and let $f \in \mathcal{F}(M, P)$. Then $\pi_1\mathcal{O}(f, \partial M) \in \mathcal{G}$, where $\mathcal{G}$ is a minimal class of groups satisfying the following conditions:*

1) $1 \in \mathcal{G}$;
2) if $A, B \in \mathcal{G}$, then $A \times B \in \mathcal{G}$;
3 if $A \in \mathcal{G}$ and $n \geq 1$, then $A \wr_n \mathbb{Z} \in \mathcal{G}$.

Definition 2. Let G, H be groups, $m \in \mathbb{Z}$ and $\gamma : H \to H$ be automorphism of order 2. Define the automorphism $\phi : G^{2m} \times H^m \to G^{2m} \times H^m$ by the formula

$$\phi(g_0, \ldots, g_{2m-1}, h_0, \ldots, h_{m-1}) = (g_{2m-1}, g_0, \ldots, g_{2m-2}, h_1, h_2, \ldots, h_{m-1}, \gamma(h_0)).$$

This automorphism ϕ generates homomorphism $\phi' : \mathbb{Z} \to G^{2m} \times H^m$. The corresponding semidirect product $G^{2m} \times H^m \rtimes_{\phi'} \mathbb{Z}$ will be denoted $(G, H) \wr_{\gamma,m} \mathbb{Z}$.

Theorem 3. [2] *Let M be a Möbius band. Then for every $f \in \mathcal{F}(M, P)$ either*

(1) exist groups $A, G, H \in \mathcal{G}$, an authmorphism $\gamma : H \to H$ of order 2 and $m \geq 1$, such that

$$\pi_1\mathcal{O}(f, \partial M) \cong A \times (G, H) \wr_{\gamma,m} \mathbb{Z},$$

(2) or there exist groups $A, G \in \mathcal{G}$ and odd number $m \geq 1$ such that

$$\pi_1\mathcal{O}(f, \partial M) \cong A \times G \wr_b \mathbb{Z}.$$

Conversely, for every such tuple (A, G, m) or (A, G, H, γ, m) there exists $f \in \mathcal{F}(M, P)$ such that we have the corresponding isomorphism.

It was shown in [1] that if M has negative Euler characterictic, then fundamental groups of orbits of functions in $\mathcal{F}(M, P)$ are direct products of such groups for functions only on cylinders, disks and Möbius bands.

References

[1] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. *Topology and its Applications*, volume 282, 2020.

[2] Iryna Kuznietsova, Sergiy Maksymenko. Deformational symmetries of smooth functions on non-orientable surfaces. *Topological Methods in Nonlinear Analysis*, 1 - 43, 2025.