On some dynamical systems given in terms of a chain A_2-representation

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Let $0 < \alpha_1 < \alpha_2$, $\alpha_1\alpha_2 = \frac{1}{2}$ and

$$[a_0; a_1, a_2, \dots, a_k, \dots] = a_0 + \cfrac{1}{a_1 + \dots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k + \dots}}}$$

— continued fraction, where $a_0 \in Z_+$, $a_j > 0$ for all $j \in N$. It is well known from [1] that for any $t \in [\alpha_1; \alpha_2]$, there exists a sequence (b_n) with $b_n \in \{\alpha_1; \alpha_2\}$ for all $n \in \mathbb{N}$, such that

$$t = [0; b_1, b_2, \dots; b_n, \dots].$$

This expression is called the A_2-representation with the alphabet $\{\alpha_1; \alpha_2\}$. A countable subset of $[\alpha_1; \alpha_2]$ has two distinct A_2-representations of the form

$$[0; b_1, \dots; b_n, \alpha_1, (\alpha_1; \alpha_2)] = [0; b_1, \dots, b_n, \alpha_2, (\alpha_2; \alpha_1)],$$

where parentheses denote the period of a given continued fraction. Numbers possessing the above property are called A_2-binary. In contrast, numbers in the interval $[\alpha_1; \alpha_2]$ that are not A_2-binary and have a unique A_2-representation are termed A_2-unary. A detailed analysis of the topological and metric properties of the A_2-representation is provided in [1].

Consider left shift operator

$$T([0; a_1, a_2, ..., a_n, ...]) = [0; a_2, a_3, ..., a_{n+1}, ...].$$

From now on, we agree not to use A_2-representations with period $(\alpha_1; \alpha_2)$ for A_2-binary numbers.

Let (ξ_n) be a sequence of independent discretely distributed random variables that take values α_1 or α_2 with probabilities $\rho \in (0; 1)$ and $1 - \rho$, respectively. Let $\eta(\cdot)$ be the Lebesgue-Stieltjes measure corresponding to the distribution

$$\xi = [0; \xi_1, \xi_2, \dots, \xi_n, \dots].$$

The Lebesgue structure of the distribution of the random variable ξ was studied in [2]. The new result is the following.

Theorem 1. *The following statements are true:*

1. The dynamical system $([\alpha_1; \alpha_2]; B(\mathbb{R}) \cap [\alpha_1; \alpha_2]; T; \eta(\cdot))$ is ergodic and the transformation T is strongly mixing:

$$\lim_{n \to +\infty} \eta(T^{-n}(A) \cap B) = \eta(A)\eta(B) \quad \forall A, B \in B(\mathbb{R}) \cap [\alpha_1; \alpha_2];$$

2. The entropy of $([\alpha_1; \alpha_2]; B(\mathbb{R}) \cap [\alpha_1; \alpha_2]; T; \eta(\cdot))$ is equal to

$$h(T) = 2 \ln \left(\frac{\rho\alpha_1 + (1 - \rho)\alpha_2 + \sqrt{(\rho\alpha_1 + (1 - \rho)\alpha_2)^2 + 4}}{2} \right).$$

References

[1] Sergey Dmytrenko., Dmitry Kyurchev., Mykola Pratsiovytyi. Continued fraction A_2-representation of real numbers and its geometry. *Ukr. Math. J*, 61(4): 452–463, 2009. (in Ukrainian)

[2] Mykola Pratsiovytyi., Dmitry Kyurchev. Properties of the distribution of the random variable defined by A_2-continued fraction with independent elements. *Random Oper. Stochastic Equations*, 17(1), 91–101, 2009.