Equivalence groupoids and group classification of (1+3)-dimensional nonlinear Schrödinger equations

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The study of Lie symmetries of nonlinear Schrödinger equations was started in the late 1970es and was then continued by many scientists, see [1, 2, 3] and references therein. An important class $\mathcal{V}$ of $(1+n)$-dimensional nonlinear Schrödinger equations with modular nonlinearities and complex-valued potentials was comprehensively considered within the framework of Lie symmetries in the literature, but the main results in this direction were obtained initially in [3] and then in [1]. The above equations are the form

$$i\psi_t + \psi_{aa} + f(\rho)\psi + V(t,x)\psi = 0, \tag{1}$$

where t and $x = (x_1, \dots, x_n)$ are the real independent variables, $n \in \mathbb{N}$, ψ is the unknown complex-valued function of (t,x), V is an arbitrary smooth complex-valued potential depending on (t,x), and f is an arbitrary complex-valued nonlinearity depending only on $\rho := |\psi|$, $f_\rho \neq 0$. Subscripts of functions denote differentiation with respect to the corresponding variables. The index a runs from 1 to n, and summation over repeated indices is assumed. Particularly known equations from the class $\mathcal{V}$ are cubic Schrödinger equations with potentials, where $f(\rho) = \rho^2$. The complete group classification of this class was carried out in [3] for $n = 1$ and in [1] for $n = 2$. Moreover, the last reference also contains preliminary results on group analysis of the class $\mathcal{V}$ for the case of arbitrary n. At the same time, even in the most physically relevant case $n = 3$, the problem of complete group classification of the class $\mathcal{V}$ is still open. The class $\mathcal{V}$ is not normalized, but it can be partitioned into three disjoint normalized subclasses, which are not related to each other by point transformations. These are the subclasses with logarithmic, power and general nonlinearities. We started extending the results of [1, 3] to the case $n = 3$ and were able to carry out the major part of the group classification for $n = 3$ and general modular nonlinearities.

Denote by $\mathcal{V}^f$ the subclass of the class $\mathcal{V}$ with $n = 3$ and a fixed general value of the nonlinearity f, i.e., $\rho f_{\rho\rho}/f_\rho$ is not a real constant, by ψ^* the complex conjugate of ψ,

$$D(1) := \partial_t, \quad J_1 := x_2\partial_3 - x_3\partial_2, \quad J_2 := x_3\partial_1 - x_1\partial_3, \quad J_3 := x_1\partial_2 - x_2\partial_1,$$

$$P(\boldsymbol{\chi}) := \chi^a\partial_a + \frac{1}{2}\chi^a_t x_a M, \quad M := i\psi\partial_\psi - i\psi^*\partial_{\psi^*},$$

where the parameters χ^a and σ are real-valued smooth functions of t.

Lemma 1. *The class $\mathcal{V}^f$ is normalized. The maximal Lie invariance algebra $\mathfrak{g}_V$ of an equation $\mathcal{L}_V$ from this class consists of the vector fields of the form $cD(1) - \kappa_a J_a + P(\boldsymbol{\chi}) + \sigma M$, where c and κ_a are arbitrary real constants and the parameter functions χ^a and σ are arbitrary real-valued smooth functions of t that satisfy the condition*

$$cV_t + (\kappa_2 x_3 - \kappa_3 x_2 + \chi^1)V_1 + (\kappa_3 x_1 - \kappa_1 x_3 + \chi^2)V_2 + (\kappa_1 x_2 - \kappa_2 x_1 + \chi^3)V_3 = \frac{1}{2}\chi^a_{tt}x_a + \sigma_t. \tag{2}$$

The kernel Lie invariance algebra of equations from the class $\mathcal{V}^f$ is $\mathfrak{g}^\cap_{\mathcal{V}^f} := \langle M\rangle$.

Any vector field of the general form presented in Lemma 1, where at least one of the parameters c, κ_a and χ^a takes a nonzero value, belongs to $\mathfrak g_V$ for a potential V satisfying the classifying condition (2) for this vector field. This is why we have $\mathfrak g_{\langle\rangle}:=\sum_V\mathfrak g_V=\langle\, D(1),\, J_a,\, P(\boldsymbol\chi),\, \sigma M\,\rangle$, where the parameter functions χ^a and σ run through the set of real-valued smooth functions of t,

A subalgebra $\mathfrak s$ of $\mathfrak g_{\langle\rangle}$ is called *appropriate* if $\mathfrak s=\mathfrak g_V$ for some V. For each of such subalgebras, we define five nonnegative integers that depend on V, are invariant under equivalence transformations of the class $\mathcal V^f$ and label the cases of Lie-symmetry extensions within this class,

$$
\begin{aligned}
&r_1:=\operatorname{rank}\{\chi\mid\exists\,\sigma\colon P(\boldsymbol\chi)+\sigma M\in\mathfrak s\},\quad k_0:=\dim\mathfrak s\cap\langle\sigma M\rangle=\dim\mathfrak g^{\cap}=1,\\
&k_1:=\dim\mathfrak s\cap\langle P(\boldsymbol\chi),\,\sigma M\rangle-k_0,\quad k_2:=\dim\mathfrak s\cap\langle J_1,\,J_2,\,J_3,\,P(\boldsymbol\chi),\,\sigma M\rangle-k_1-k_0,\\
&k_3:=\dim\mathfrak s-k_2-k_1-k_0.
\end{aligned}
$$

One has $r_1\in\{0,1,2,3\}$, $k_2\in\{0,1,3\}$, $r_1\leqslant k_1$, $k_1\in\{0,\dots,6\}$ and $k_3\in\{0,1\}$ [1, Section 6].

The following lemmas are useful in the course of the group classification of the class $\mathcal V^f$ with $n=3$.

Lemma 2. *(i) If $k_2=3$ and $Q^0=P(\boldsymbol\chi^0)+\sigma^0M+\zeta^0I\in\mathfrak s$, then $P(\chi^{0a}\boldsymbol\delta_b)+\breve\sigma^{ab}M\in\mathfrak s$ for any $a,b\in\{1,2,3\}$ and some functions $\breve\sigma^{ab}$ of t.*

(ii) If $k_2=3$, then $\mathfrak s\supseteq\langle J_1,\,J_2,\,J_3\rangle$ modulo the point equivalence in the class $\mathcal V^f$, and $r_1\in\{0,3\}$.

Lemma 3. *If $\boldsymbol\chi$ and $\breve{\boldsymbol\chi}$ are linearly independent and $\boldsymbol\chi\cdot\breve{\boldsymbol\chi}_t-\boldsymbol\chi_t\cdot\breve{\boldsymbol\chi}=0$, then* $\operatorname{rank}(\boldsymbol\chi,\breve{\boldsymbol\chi})=2$.

Lemma 4. *If* $\operatorname{rank}(\boldsymbol\chi^1,\boldsymbol\chi^2)=\operatorname{rank}(\boldsymbol\chi^1,\boldsymbol\chi^2,\boldsymbol\chi)=2$ *and $\boldsymbol\chi\cdot\boldsymbol\chi^l-\boldsymbol\chi_t\cdot\boldsymbol\chi^l=0$, $l=1,2$, then $\boldsymbol\chi\in\langle\boldsymbol\chi^1,\boldsymbol\chi^2\rangle$.*

Lemma 5. *Let $r_1=2$, i.e., the algebra $\mathfrak g_V$ contains at least two vector fields of the form $Q^s=P(\boldsymbol\chi^s)+\sigma^sM+\zeta^sI$, $s=1,2$, where* $\operatorname{rank}(\boldsymbol\chi^1,\boldsymbol\chi^2)=2$ *for any t in the related interval. Denote $\boldsymbol\chi^0:=\boldsymbol\chi^1\times\boldsymbol\chi^2\neq\mathbf 0$. Then $\boldsymbol\chi^1\cdot\boldsymbol\chi^2_t-\boldsymbol\chi^1_t\cdot\boldsymbol\chi^2=\mathrm{const}$ and the following holds:*

(i) $k_1\in\{2,3,4\}$.

(ii) $k_1=2$ if and only if $\big(\boldsymbol\chi^0\times\boldsymbol\chi^0_t,\,\boldsymbol\chi^0_{tt}+\boldsymbol\chi^1_t\times\boldsymbol\chi^2_t\big)\neq0$.

(iii) $k_1=3$ if and only if $\big(\boldsymbol\chi^0\times\boldsymbol\chi^0_t,\,\boldsymbol\chi^0_{tt}+\boldsymbol\chi^1_t\times\boldsymbol\chi^2_t\big)=0$ but $\boldsymbol\chi^0\times\boldsymbol\chi^0_t\neq\mathbf 0$.

(iv) $k_1=4$ if and only if $\boldsymbol\chi^0\times\boldsymbol\chi^0_t=\mathbf 0$.

We have classified the equations from the class $\mathcal V^f$ with $r_1\in\{0,1,3\}$ and are almost complete classification for the case $r_1=2$. In particular, if $r_1=3$, $k_2=0$, $k_3=1$, then up to the point equivalence within the class $\mathcal V^f$, the algebra $\mathfrak g_V$ necessary contains the vector field $D(1)+\kappa J_3$ with $\kappa=\mathrm{const}$. For nonzero κ, the corresponding case of Lie symmetry extension is the following:

$$
\begin{aligned}
&V=\tfrac14(\alpha_1\omega_1^2+\alpha_2\omega_2^2+\alpha_3x_3^2)+\tfrac12(\beta_1\omega_1+\beta_2\omega_2)x_3+i\nu\colon\\
&\mathfrak g_V=\langle M,\,P(\theta^{p1}\cos\kappa t-\theta^{p2}\sin\kappa t,\,\theta^{p1}\sin\kappa t+\theta^{p2}\cos\kappa t,\,\theta^{p3}),\,p=1,\dots,6,\,D(1)+\kappa J_3\rangle,
\end{aligned}
$$

where $\omega_1:=x_1\cos\kappa t+x_2\sin\kappa t$, $\omega_2:=-x_1\sin\kappa t+x_2\cos\kappa t$, $\omega_3:=x_3$; α_1, α_2, β_1, β_2 and κ are real constants with $\alpha_2\neq\alpha_1\neq0$, and $\kappa\neq0$; $(\theta^{p1}(t),\theta^{p2}(t),\theta^{p3}(t))$ are linearly independent solutions of the system,

$$
\theta^1_{tt}-2\kappa\theta^2_t=(\kappa^2+\alpha_1)\theta^1+\beta_1\theta^3,\quad \theta^2_{tt}+2\kappa\theta^1_t=(\kappa^2+\alpha_2)\theta^2+\beta_2\theta^3,\quad \theta^3_{tt}=\alpha_3\theta^3.
$$

References

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