The Arithmetic and Geometric Properties of Rational Surfaces

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Rational surfaces are an important research subject in surface theory. The surfaces with minimal degree and Del Pezzo surfaces are classical examples of rational surfaces. In this talk, we will focus on recent advances in the study of rational surfaces. In particular, we will investigate the structure of rational surfaces from both arithmetic and geometric perspectives.

Theory of Galois covers is the geometric counterpart of classical Galois theory. In [1], we proved that the Galois covers of surfaces with minimal degree are simply connected surfaces. Furthermore, we also considered extending these results to Zappatic surfaces. We proved that:

Theorem 1. *If a smooth projective algebraic surface deforms to a Zappatic surface of type E_n, $n \geq 4$, then its Galois cover is simply connected and of general type.*

Furthermore, we have the following result:

Theorem 2. *For a complex algebraic surface $X \subset \mathbb{P}^2$, if there exists a degeneration X_0 such that X_0 has only singularities of type R_k, and X_0 has R_k loops, then the Galois cover of the surface X is not simply connected.*

The topological invariants of a surface are important invariants in surface theory. In this talk, we will also introduce the invariants of Galois covers, with a particular focus on Zappatic surfaces that have only singularities of type R_k:

Theorem 3. *The signature $\tau(X_{Gal}) = \frac{1}{3}n!\ (-I(T))$ where n is the degree of X, and $I(T)$ is the number of vertices of degree 2 in the dual graph T of X_0.*

We also extend this result to surface fibrations over rational surfaces, thereby obtaining some geometric properties of rational surfaces:

First, for elliptic fibrations, we have ([2]):

Theorem 4. *Given an rational elliptic surface S over $\mathbb{P}^1$, with the generic fibre F , we give the number of integral sections.*

Secondly, for fibrations with two singular fibers, we have ([3]).

Theorem 5. *Let $f : S \longrightarrow \mathbb{P}^1$ be a relatively minimal fibration of genus $g \geq 2$ with two singular fibers, F_1 and F_2.*

- *If $0 \neq \tau(S)$, then $\tau(S) \leq -4$.*
- *If $\tau(S) < -4$, then $\tau(S) \leq -6$.*

Finally, we will present some applications of the theorems discussed and propose several open problems.

References

[1] M.Amram, C. Gong, J-L Mo.*On the Galois covers of degenerations of surfaces of minimal degree*, *Mathematischen Nachrichten*, 2023, 296, 1351–1365.

[2] J-L Mo. *A note on a question of Shioda about integral sections*, *Algebra Colloquium*, accepted.

[3] C.Gong, J-L M. *On the topological index of a singular fiber in a family of curves*, *Bulletin of the Korean Mathematical Society*, accepted.