Surfaces with flat normal connection in 4-dimensional space forms

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Let $N$ be a 4-dimensional Riemannian space form with constant sectional curvature $L_0$. Let $h$ be the metric of $N$ and $\nabla$ the Levi-Civita connection of $h$. Let $M$ be a Riemann surface and $F : M \longrightarrow N$ a conformal immersion. Let $(u, v)$ be local isothermal coordinates of $M$. Then the induced metric of $M$ by $F$ is represented as $g = e^{2\lambda}(du^2 + dv^2)$ for a function $\lambda$. We set $T_1 := dF(\partial/\partial u)$, $T_2 := dF(\partial/\partial v)$. Let $N_1$, $N_2$ be normal vector fields of $F$ satisfying $h(N_1, N_1) = h(N_2, N_2) = e^{2\lambda}$, $h(N_1, N_2) = 0$.

Suppose that $N$ is oriented and that $(T_1, T_2, N_1, N_2)$ gives the orientation. We set

$$e_1 := \frac{1}{e^\lambda}T_1, \quad e_2 := \frac{1}{e^\lambda}T_2, \quad e_3 := \frac{1}{e^\lambda}N_1, \quad e_4 := \frac{1}{e^\lambda}N_2$$

and

$$\begin{aligned}&\Omega_{\pm,1} := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4), \quad \Omega_{\pm,2} := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2),\\ &\Omega_{\pm,3} := \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).\end{aligned}$$

The two-fold exterior power $\bigwedge^2 F^*TN$ of the pull-back bundle $F^*TN$ on $M$ by $F$ is of rank 6 and decomposed into two subbundles $\bigwedge^2_\pm F^*TN$ of rank 3, and $\Omega_{\pm,1}$, $\Omega_{\pm,2}$, $\Omega_{\pm,3}$ form local orthonormal frame fields of $\bigwedge^2_\pm F^*TN$ respectively.

Let $\hat{\nabla}$ be the connection of $\bigwedge^2 F^*TN$ induced by $\nabla$. Then $\hat{\nabla}$ gives connections of $\bigwedge^2_\pm F^*TN$ and we obtain

$$\begin{aligned}\hat{\nabla}_{T_1}(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3}) &= (\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})\begin{bmatrix} 0 & -W_\pm & -Y_\mp \\ W_\pm & 0 & \pm\psi_\pm \\ Y_\mp & \mp\psi_\pm & 0 \end{bmatrix},\\ \hat{\nabla}_{T_2}(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3}) &= (\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})\begin{bmatrix} 0 & \mp Z_\pm & \pm X_\mp \\ \pm Z_\pm & 0 & \mp\phi_\mp \\ \mp X_\mp & \pm\phi_\mp & 0 \end{bmatrix}\end{aligned} \tag{1}$$

([4]), where

(i) $W_\pm$, $X_\pm$, $Y_\pm$, $Z_\pm$ are functions related to the second fundamental form $\sigma$ of $F$ satisfying $W_+ + W_- = X_+ + X_-$, $Y_+ + Y_- = Z_+ + Z_-$,

(ii) $\phi_\pm := \lambda_u \mp \mu_2$, $\psi_\pm := \lambda_v \mp \mu_1$, and $\mu_1$, $\mu_2$ are functions related to the normal connection $\nabla^\perp$ of the immersion $F$ (in particular, if $\nabla^\perp$ is flat, then there exists a function $\gamma$ satisfying $\gamma_u = \mu_1$, $\gamma_v = \mu_2$).

Let $\hat{R}$ be the curvature tensor of $\hat{\nabla}$. Then computing $\hat{R}(T_1, T_2)(\Omega_{\pm,1}\ \Omega_{\pm,2}\ \Omega_{\pm,3})$ by (1) and noticing that $N$ is a space form of constant sectional curvature $L_0$, we obtain

$$W_\mp X_\pm + Y_\pm Z_\mp = L_0 e^{2\lambda} + (\phi_\pm)_u + (\psi_\mp)_v \tag{2}$$

and

$$\begin{aligned}(Y_\pm)_v \mp (X_\pm)_u &= \pm W_\mp\phi_\pm - Z_\mp\psi_\mp,\\ (W_\mp)_v \pm (Z_\mp)_u &= \mp Y_\pm\phi_\pm - X_\pm\psi_\mp.\end{aligned} \tag{3}$$

As in [5], (2) is equivalent to the system of the equations of Gauss and Ricci, and (3) is equivalent to the system of the equations of Codazzi.

In [5], immersions with flat normal connection are studied. Let $R^{\perp}$ be the curvature of the normal connection $\nabla^{\perp}$. By definition, $R^{\perp}=0$ just means that $\nabla^{\perp}$ is flat. If $F$ has a parallel normal vector field, then the second fundamental form $\sigma$ satisfies the linearly dependent condition and then $\nabla^{\perp}$ is flat (see [5]). Suppose that the curvature $K$ of $g$ is nowhere equal to $L_0$. Then $F$ has a parallel normal vector field if and only if $\sigma$ satisfies the linearly dependent condition ([5]). On the other hand, if we suppose $K=L_0$, then the linearly dependent condition of $\sigma$ does not necessarily mean the existence of parallel normal vector fields ([5]).

By (2), $F$ satisfies $W_{\mp}X_{\pm}+Y_{\pm}Z_{\mp}=0$ if and only if $R^{\perp}=0$ and $K=L_0$ hold. Suppose that there exist functions $k_{\pm}$ satisfying

$$(W_{\mp}, Z_{\mp}) = k_{\pm}(-Y_{\pm}, X_{\pm}). \tag{4}$$

Then $W_{\mp}X_{\pm}+Y_{\pm}Z_{\mp}=0$ hold. Applying (4) to (3), we see that there exist functions $f_{\pm}$ satisfying

$$X_{\pm} = \pm\frac{(f_{\pm})_v}{\sqrt{1+k_{\pm}^2}}, \quad Y_{\pm} = \frac{(f_{\pm})_u}{\sqrt{1+k_{\pm}^2}}, \tag{5}$$

and by the equation of Ricci, we obtain $(f_+)_u^2+(f_+)_v^2=(f_-)_u^2+(f_-)_v^2$. Therefore, if we suppose $(f_+)_u^2+(f_+)_v^2\neq 0$, then there exists a function $\psi$ satisfying

$$\begin{bmatrix} (f_-)_u \\ (f_-)_v \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} (f_+)_v \\ (f_+)_u \end{bmatrix}. \tag{6}$$

Suppose that $k_{\pm}$ is nowhere zero and that $X_{\pm}$, $Y_{\pm}$ satisfy $X_+^2Y_-^2-X_-^2Y_+^2\neq 0$. Then the second fundamental form $\sigma$ does not satisfy the linearly dependent condition ([5]), and using (3), (4) and (5), we can obtain an over-determined system for the function $\gamma$ related to $\nabla^{\perp}$ ([5]). In addition, if we suppose $L_0=0$, then the compatibility condition of this over-determined system can be represented as an over-determined system of polynomial type with degree two for the function $\psi$ in (6) ([5]). See [1] for over-determined systems of polynomial type.

In the above discussions, we supposed that $N$ is a Riemannian space form. Suppose that $N$ is a 4-dimensional neutral space form with constant sectional curvature $L_0$. Then for a Riemann or Lorentz surface $M$ and a space-like or time-like, and conformal immersion $F: M \longrightarrow N$, we can have similar discussions and obtain analogous results ([4], [5]). See [2], [3] for time-like surfaces in $N$ with zero mean curvature vector and $K\equiv L_0$ (such surfaces have flat normal connection). In the case where $N$ is a 4-dimensional Lorentzian space form with constant sectional curvature $L_0$, noticing that the complex bundle $\bigwedge^2 F^*TN\otimes \boldsymbol{C}$ is decomposed into two subbundles of complex rank 3, we can have similar discussions and obtain analogous results ([4], [5]).

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## References


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