Wilson loops as a device for studying phase transitions and conductivity effects

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Wilson loops have become a fundamental tool in high-energy physics. They have allowed the formulation and verification of key properties of quantum chromodynamics and laid the foundation for the numerical analysis of strong interactions. Their applications extend beyond QCD to broad areas of theoretical physics, including string theory and quantum gravity [1].

The importance of Wilson loops can be summarized in the following list:

• Proof of confinement – Wilson loops allowed us to quantitatively explain why quarks are not observed in a free state;

• Development of lattice QCD – this approach made it possible to simulate strong interactions on computer;

• Application in string theory – Wilson loops are associated with the so-called "string break" effects and the formation of "color tubes" between quarks;

• Generalizations to other theories used in various gauge theories, including gravity and condensed matter.

There are two most important aspects of the application of Wilson loops that need to be addressed:
1. The topological aspect:
In systems with topological order, the conductivity can be related to topological invariants that are expressed in terms of Wilson loops. For example, in the quantum Hall effect, the conductivity is quantized, which can be described using nontrivial topological configurations of Wilson loops.
2. Lattice QCD:
In lattice QCD calculations, Wilson loops are used to study the potentials between quarks and the properties of quark-gluon plasma. The plasma conductivity depends on the mobility of quarks and gluons, whose interactions are encoded in the Wilson loops.

For a quantized magnetic field, $\Phi = 2\pi n/e$ the Wilson loop takes the value

$$W(c) = e^{i2\pi n}.$$

For a magnetic monopole with a magnetic charge g at the center of the field of radius R the Wilson loop takes the value

$$W(C) = e^{2\pi g/R}.$$

Quantization of magnetic flux leads to discrete values of the Wilson loop, and quantization o magnetic charge indicates the topological nature of the monopole.

A powerful tool that combines physics and mathematics, Chern-Simons theory allows us to study the topological properties of systems and their stability [2]. Chern-Simons theory describes topological invariants that do not depend on the metric of the space, but only on its topological structure. The basis of the theory is the Chern-Simons action, which is defined in three-dimensional space and depends on the gauge field. It has the form:

$$S = \frac{k}{4\pi}\int tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

average value of Wilson loop

$$< W_R(C) >= \frac{1}{Z}\int DAW_R(C)e^{iS(A)},$$

where $Z = \int DAe^{iS(A)}$ is a statistical sum.

Chern-Simons theory plays a key role in describing topological phases of matter, such as the fractional quantum Hall effect (FQHE) [3]. The Laughlin states are the most well-known examples of fractional quantum Hall states, occurring at filling factors $\nu = 1/m$, where m is an odd integer (e.g., $\nu = 1/3, 1/5, \ldots$). In deriving the topological response of the Laughlin state, particularly the quantized Hall conductivity let's obtain the effective action for the external electromagnetic field A_μ connected with the Lagrangian

$$L = \frac{m}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \frac{e}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a_\rho.$$

To derive the Hall conductivity from the effective action, we start with the effective Chern-Simons action for the external electromagnetic field A_μ:

$$L_{eff} = \frac{e^2}{4\pi m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho.$$

This action describes the topological response of the system to the external field A_μ. The coefficient $\frac{e^2}{4\pi m}$ encodes the filling factor $\nu = 1/m$ of the Laughlin state. Using Hall conductivity expression $\sigma_{xy} = \frac{j^x}{E_y}$ and substituting $j^\mu = \frac{\delta L_{eff}}{\delta A_\mu}$, we get

$$\sigma = \frac{e^2}{4\pi m} = \frac{e^2}{h}\nu.$$

This derivation shows how the topological Chern-Simons term in the effective action leads to the quantized Hall conductivity, a hallmark of the Laughlin state.

To find the phase factor for exchanging two quasiparticles in the Laughlin state, we need to analyze the braiding statistics of quasiparticles using the framework of Chern-Simons theory and Wilson loops. Consider two quasiparticles at positions r_1 and r_2. When the quasiparticles are exchanged, their trajectories form a braid in spacetime. The phase factor associated with this exchange can be computed using the linking number of the Wilson loops $< W(C_1)W(C_2) >= e^{i\frac{2\pi}{m}Link(C_1,C_2)}$. For a simple exchange of two quasiparticles, the linking number is $Link(C_1, C_2) = 1$ and

$$< W(C_1)W(C_2) >= e^{i\frac{2\pi}{m}}.$$

Exchanging two quasiparticles corresponds to half of a full braid and we have

$$e^{i\frac{\pi}{m}}.$$

The phase factor signals about:
1. The Chern-Simons term in the effective action enforces the fractional braiding phase.
2. The exchange of two quasiparticles corresponds to a π rotation in spacetime, leading to the phase factor $e^{i\pi/m}$.
3. The linking number of Wilson loops describing quasiparticle trajectories.

From the results of comparing the expressions for the Wilson loop and conductivity, we see proportionality of the average value of Wilson loop to the filling factor $\nu = 1/m$. The movement of one quasiparticle around another produces a topological phase shift associated with the charge and statistics of the particles.

References

[1] H. Eschrig. Topology and Geometry for Physics. *Lecture Notes in Physics*: 220–222, 2011.
[2] S.-S. Chern, J. Simons. Characteristic forms and geometric invariants. *Annals of Mathematics*: volume 99(1), 48–69, 1974.
[3] M. Greiter. Microscopic formulation of the hierarchy of quantized Hall states. *Physics Letters B*: volume 336(1) 48–53, 1994.