Translation length formula for two-generated groups acting on trees

Kamil Orzechowski
(University of Rzeszów, Poland)
E-mail: kamilo@dokt.ur.edu.pl

The notion of a *tree* is considered at various levels of generality in graph theory, geometry, and topology. A combinatorial tree (i.e., a connected circuit-free graph) determines an integer-valued metric on its set of vertices; the resulting metric space is called a $\mathbb{Z}$-*tree*. A metric space that is geodesic and uniquely arcwise-connected is called an $\mathbb{R}$-*tree*. These examples can be generalized by defining the notion of a Λ-*tree* [1, Ch. 2, §1], where Λ is any totally ordered Abelian group. A Λ-tree (X, d) is a Λ-metric space (i.e., its metric takes values in Λ instead of $\mathbb{R}$) satisfying certain natural conditions reflecting the tree-like structure of X.

All isometries of a Λ-tree (X, d) onto itself can be divided into three types: elliptic, hyperbolic, and inversions [1, Ch. 3, §1]. Let g be an isometry of a Λ-tree (X, d). It is called *elliptic* if it has a fixed point in X; g is called an *inversion* if g has no fixed points in X but g^2 does; otherwise g is called *hyperbolic*. The *translation length* of g [5, p. 297] is defined as

$$\|g\| := \begin{cases} 0 & \text{if } g \text{ is an inversion,} \\ \min\{d(x, gx) \colon x \in X\} & \text{otherwise.} \end{cases} \tag{1}$$

In fact, if g is not an inversion, the set of points for which the minimum in (1) is reached is a nonempty closed subtree of X. Hyperbolic isometries are precisely those with $\|g\| > 0$. If g is hyperbolic, then the set $\{x \in X \colon d(x, gx) = \|g\|\}$ is called the *axis* of g; it is isometric to a convex subset of Λ and the action of g on its axis corresponds to the translation by $\|g\|$, which justifies the terminology.

Parry [5] proved that a function $\|\cdot\| \colon G \to \Lambda_+$ is the translation length function for some action of a group G on a Λ-tree if and only if it satisfies a certain set of algebraic conditions; such a function is called a *pseudo-length* on G.

Our main result concerns an explicit formula for $\|g\|$, $g \in \langle a, b\rangle$, in the case of a pair $(a, b) \in G \times G$ satisfying the conditions

$$\|a\| > 0, \quad \|b\| > 0, \quad |\|a\| - \|b\|| < \min\{\|ab\|, \|ab^{-1}\|\}. \tag{2}$$

We call such a pair $(a, b) \in G \times G$ a *ping-pong pair.*

Theorem 1. *If $\|\cdot\|$ is a pseudo-length on a group G and $a, b \in G$ satisfy (2), then*

$$2\|w\| = \left(\sum_{i=1}^{n-1} \|x_i x_{i+1}\|\right) + \|x_n x_1\| > 0,$$

for any cyclically reduced word $w = x_1 \dots x_n$, $x_i \in \{a, b, a^{-1}, b^{-1}\}$, $n \geq 1$.

An important consequence of Theorem 1 is the fact that if G acts by isometries on a Λ-tree (X, d) with the translation length function $\|\cdot\|$, and $(a, b) \in G \times G$ is a ping-pong pair with respect to $\|\cdot\|$, then the subgroup $\langle a, b\rangle \leq G$ is free of rank two and acts freely, without inversions, and properly discontinuously on (X, d). This result is known, see [2, Propositions 1 and 2]. The cited proofs are geometric in nature and rely on drawing pictures or "ping-pong" type arguments. We present a combinatorial approach, using only the defining conditions of a pseudo-length and not referring to any geometric interpretation.

Our other result is the existence and uniqueness of a pseudo-length $\|\cdot\| \colon F(a, b) \to \Lambda_+$ on the free group $F(a, b)$ under certain conditions imposed on the values it takes at a, b, ab, and ab^{-1}.

Theorem 2. *Let $\alpha, \beta, \gamma, \delta \in \Lambda$ be such that*

$$\begin{gathered}\gamma - \alpha - \beta \in 2\Lambda, \quad \delta - \alpha - \beta \in 2\Lambda; \\ \textit{either } \gamma = \delta > \alpha + \beta \quad \textit{or } \max\{\gamma, \delta\} = \alpha + \beta; \\ \alpha > 0,\ \beta > 0,\ |\alpha - \beta| < \min\{\gamma, \delta\}.\end{gathered} \tag{3}$$

There exists exactly one pseudo-length $\|\cdot\| \colon F(a,b) \to \Lambda_+$ such that $\|a\| = \alpha$, $\|b\| = \beta$, $\|ab\| = \gamma$, and $\|ab^{-1}\| = \delta$.

Finally, we use Theorem 2 to prove that, in the case of a subgroup $\Lambda \leq \mathbb{R}$, any *purely hyperbolic* (i.e., $\|g\| > 0$ for $g \neq 1$) pseudo-length on $F(a,b)$ can be described by four elements of Λ satisfying (3), and an outer automorphism of $F(a,b)$. We present an algorithm to effectively find such a description of a given purely hyperbolic pseudo-length on $F(a,b)$. The space of all these pseudo-lengths is related to the concept of the Culler–Vogtmann *outer space* [3].

References

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