C^∞-structures approach to travelling wave solutions of the gKdV equation

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In this work, we present a geometric approach to finding travelling wave solutions of the generalized Korteweg–de Vries (gKdV) equation [1]. The method is rooted in the theory of $\mathcal{C}^\infty$-structures and leverages concepts from differential geometry, particularly the theory of distributions and Pfaffian systems (see [2]).

The gKdV equation, given by

$$u_t + u_{xxx} + a(u)u_x = 0, \tag{1}$$

where $u = u(t, x)$ and $a(u)$ is a smooth function, is an important equation in mathematics and physics, with the standard Korteweg–de Vries equation being a well-known special case. The function $a(u)$ plays a crucial role in determining the specific characteristics of the gKdV equation and its applicability to different physical scenarios, by establishing the nature and strength of the nonlinearity in the system.

The core of this talk lies in the application of the C^∞-structure-based method to integrate the ordinary differential equation (ODE) obtained from the travelling wave ansatz applied to (1), i.e., the equation

$$-cy' + y''' + a(y)y' = 0, \tag{2}$$

where $y = y(z)$ and $c \in \mathbb{R}$.

Roughly speaking, a $\mathcal{C}^\infty$-structure for an mth-order ODE is an ordered collection of m vector fields giving rise to a sequence of involutive distributions, starting with the distribution generated by the vector field associated to the ODE. The key geometric insight is that the integral manifolds of these distributions contain the prolongation of the solutions of the ODE. The method enables the integration by transforming the problem into a sequence of m completely integrable Pfaffian equations.

To apply this geometric method to equation (2), we first construct a $\mathcal{C}^\infty$-structure for this ODE, starting with the Lie symmetry ∂_z of the equation. The $\mathcal{C}^\infty$-structure-based integration algorithm is then applied, leading to a sequence of three Pfaffian equations. At the final step we obtain an implicit general solution for the travelling wave solutions of the gKdV equation.

Finally, we present explicit solutions for the gKdV equation in various cases, depending on the choice of the function $a(u)$: the modified KdV equation, a family of KdV equations with power-law nonlinearity, and the Schamel–Korteweg–de Vries equation.

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References

[1] M. Tsutsumi, T. Mukasa, and R Iino. On the generalized Korteweg-deVries equation. *Proc. Japan Acad.*, 46(9):921 – 925, 1970.

[2] A. J. Pan-Collantes, A. Ruiz, C. Muriel, and J. L. Romero. $\mathcal{C}^\infty$-symmetries of distributions and integrability. *J. Diff. Eq.*, 348:126–153, 2023.