On E-endomorphisms of the upper subsemigroup of the bicyclic monoid

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We shall follow the terminology of [1]. By ω we denote the set of all non-negative integers.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq=1$. Any element of $\mathscr{C}(p,q)$ has the unique representation b^ia^j, $i,j\in\omega$. In [3] the following anti-isomorphic subsemigroups of the bicyclic monoid

$$\mathscr{C}_+(a,b)=\left\{b^ia^j\in\mathscr{C}(a,b)\colon i\leqslant j,\, i,j\in\omega\right\}$$

and

$$\mathscr{C}_-(a,b)=\left\{b^ia^j\in\mathscr{C}(a,b)\colon i\geqslant j,\, i,j\in\omega\right\}$$

are studied. All injective endomorphism of $\mathscr{C}_+(a,b)$ are described in [2].

Let S be a semigroup with the non-empty set of idempotent $E(S)$. An endomorphism α of S is said to be *E-endomorphism* if $(s)\alpha\in E(S)$ for all $s\in S$.

Theorem 1. *Let α be a monoid endomorphism of the semigroup $\mathscr{C}_+(a,b)$. Then the following conditions are equivalent:*

(1) *α is an E-endomorphism;*
(2) *there exists a non-idempotent element b^ia^j of $\mathscr{C}_+(a,b)$ such that $(b^ia^j)\alpha$ is an idempotent of $\mathscr{C}_+(a,b)$;*
(3) *the image $(\mathscr{C}_+(a,b))\alpha$ is a finite subset of $\mathscr{C}_+(a,b)$.*

By $\omega_{\max}$ we denote the set ω with the semilattice operation $n\cdot m=\max\{n,m\}$, $n,m\in\omega$. We extend the semilattice operation of $\omega_{\max}$ onto $\omega^*=\omega\cup\{\infty\}$ with $\infty\notin\omega$ in the following way

$$n\cdot\infty=\infty\cdot n=\infty\cdot\infty=\infty, \qquad \text{for all} \quad n\in\omega.$$

The set ω^* with so defined semilattice operation we denote by $\omega^*_{\max}$.

An endomorphism ε of the semilattice $\omega^*_{\max}$ is called *bounded* if there exists $n_\varepsilon\in\omega$ such that $(x)\varepsilon\leqslant n_\varepsilon$ for all $x\in\omega^*_{\max}$. It is obvious that the composition of any two bounded endomorphisms of the semilattice $\omega^*_{\max}$ is a bounded endomorphism. By $\mathfrak{End}_{\mathbf{b}}(\omega^*_{\max})$ we denote the semigroup of all bounded endomorphisms of semilattice $\omega^*_{\max}$ and by $\mathfrak{End}_{\mathbf{E}}(\mathscr{C}_+(a,b))$ the semigroup of E-endomorphisms of $\mathscr{C}_+(a,b)$.

Theorem 2. *The semigroups $\mathfrak{End}_{\mathbf{b}}(\omega^*_{\max})$ and $\mathfrak{End}_{\mathbf{E}}(\mathscr{C}_+(a,b))$ are isomorphic.*

References

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[3] S. O. Makanjuola and A. Umar, *On a certain sub semigroup of the bicyclic semigroup*, Commun. Algebra **25** (1997), no. 2, 509-519.