On existence of continuations for different types of metrics

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The problems of continuation of a partially defined metric and a partially defined ultrametric were considered in [1] and [2], respectively. Using the language of graph theory we generalize the criteria of existence of continuation obtained in these papers. For these purposes we use the concept of a triangle function introduced by M. Bessenyei and Z. Páles in [3], which gives a generalization of the triangle inequality in metric spaces. The obtained result allows us to get criteria of the existence of continuation for a wide class of semimetrics including metrics, ultrametrics, semimetrics with power triangle inequality, etc.

Let X be a nonempty set. Recall that a mapping $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$, is a *metric* if for all $x, y, z \in X$ the following axioms hold: (i) $(d(x,y) = 0) \Leftrightarrow (x = y)$, (ii) $d(x,y) = d(y,x)$, (iii) $d(x,y) \leqslant d(x,z) + d(z,y)$. The pair (X,d) is called a *metric space*. If only axioms (i) and (ii) hold then the pair (X,d) is called a *semimetric space*. We shall say that d is a *pseudosemimetric* if only axiom (ii) and condition $d(x,x) = 0$ hold. In this case the pair (X,d) will be called a *pseudosemimetric space*.

Definition 1. ([3]) Consider a pseudosemimetric space (X,d). We shall say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and monotone increasing in both of its arguments, satisfies $\Phi(0,0) = 0$ and, for all $x, y, z \in X$, the generalized triangle inequality

$$d(x,y) \leqslant \Phi(d(x,z), d(y,z))$$

holds. We also shall say that d is a Φ-pseudosemimetric if Φ is a triangle function for d.

Let $n \in \mathbb{N}$. For every triangle function Φ consider a function $\Phi^*\colon \mathbb{R}^n_+ \to \mathbb{R}^+$ of n variables, defined as

$$\Phi^*(x_1, ..., x_n) = \begin{cases} x_1, & \text{if } n = 1, \\ \Phi(x_1, x_2), & \text{if } n = 2, \\ \Phi(x_1, \Phi(x_2, \Phi(x_3, ...\Phi(x_{n-2}, \Phi(x_{n-1}, x_n))))), & \text{if } n \geqslant 3. \end{cases}$$

It is clear that Φ^* is monotone increasing in all of its variables as well as Φ.

Recall that a graph G is an ordered pair (V,E) consisting of a set $V = V(G)$ of vertices and a set $E = E(G)$ of edges. A graph $G = (V,E)$ together with a weight $w\colon E(G) \to \mathbb{R}^+$ is called a weighted graph. Let (G,w) be a weighted graph and let u, v be vertices belonging to a connected component of G. Let us denote by $\mathcal{P}_{u,v} = \mathcal{P}_{u,v}(G)$ the set of all paths joining u and v in G. For the path $P \in \mathcal{P}_{u,v}$ define the Φ-*weight* of this path by

$$w_\Phi(P) = \begin{cases} 0, & \text{if } E(P) = \varnothing, \\ \Phi^*(w(e_1), ..., w(e_n)), & \text{otherwise,} \end{cases}$$

where $e_1, ..., e_n$ are all edges of the path P. Write

$$d^w_\Phi(u,v) = \inf\{w_\Phi(P)\colon P \in \mathcal{P}_{u,v}\}.$$

In the case $\Phi(x,y) = x+y$ for the connected graph G the function d^w_Φ is a shortest-path pseudometric [1] on the set $V(G)$ and in the case $\Phi(x,y) = \max\{x,y\}$ it is a subdominant pseudoultrametric [2].

In the next lemma and further we identify a pseudosemimetric space (X,d) with the complete weighted graph $(G, w_d) = (G(X), w_d)$ having $V(G) = X$ and satisfying the equality

$$w_d(\{x,y\}) = d(x,y)$$

for every pair of different points $x, y \in X$.

Lemma 2. ([4]) *Let (X, d) be a pseudosemimetric space with the triangle function Φ. Then for every cycle $C \subseteq G(X)$ and for every $e \in E(C)$ the inequality $w_d(e) \leqslant w_\Phi(C \setminus e)$ holds, where $C \setminus e$ is a path obtained from the cycle C by the removal of the edge e.*

We are interested in the following question. Let (G, w) be a weighted graph. Does there exist a Φ-pseudosemimetric $d\colon V(G) \times V(G) \to \mathbb{R}^+$ such that the given weight $w\colon E(G) \to \mathbb{R}^+$ has a continuation to d? I.e., the equality

$$w(\{u, v\}) = d(u, v)$$

holds for all $\{u, v\} \in E(G)$. If such a continuation exists, then we say that w is a Φ-*pseudosemimetrizable weight*.

Theorem 3. ([4]) *Let (G, w) be a weighted graph and let Φ be a continuous in both variables triangle function. The following statements are equivalent.*

- *(i) The weight w is Φ-pseudosemimetrizable.*
- *(ii) The equality $w(\{u, v\}) = d_\Phi^w(u, v)$ holds for all $\{u, v\} \in E(G)$.*
- *(iii) For every cycle $C \subseteq G$ and for every $e \in C$ the inequality $w(e) \leqslant w_\Phi(C \setminus e)$ holds, where $C \setminus e$ is a path obtained from C by the removal of the edge e.*

Corollary 4. ([4]) *Let (G, w) be a weighted graph. Then the corresponding statements are equivalent.*

- *(i_1) The weight w is pseudometrizable, i.e., $\Phi(x, y) = x + y$.*
- *(i_2) For every cycle $C \subseteq G$ the following inequality holds:*

$$2 \max_{e \in E(C)} w(e) \leqslant \sum_{e \in C} w(e).$$

- *(ii_1) The weight w is pseudoultrametrizable, i.e., $\Phi(x, y) = \max\{x, y\}$.*
- *(ii_2) For every cycle $C \subseteq G$ there exist at least two different edges $e_1, e_2 \in E(C)$ such that*

$$w(e_1) = w(e_2) = \max_{e \in E(C)} w(e).$$

- *(iii_1) The weight w is Φ-pseudosemimetrizable with $\Phi(x, y) = (x^p + y^p)^{\frac{1}{p}}$, $p > 0$.*
- *(iii_2) For every cycle $C \subseteq G$ and every $e \in C$ the following inequality holds:*

$$w(e) \leqslant \left(\sum_{\tilde{e} \in C \setminus e} w^p(\tilde{e}) \right)^{\frac{1}{p}}.$$

- *(iv_1) The weight w is Φ-pseudosemimetrizable with $\Phi(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y))$, where $\varphi\colon [0, \infty) \to [0, \infty)$ is a homeomorphism.*
- *(iv_2) For every cycle $C \subseteq G$ and every $e \in C$ the following inequality holds:*

$$w(e) \leqslant \varphi^{-1} \left(\sum_{\tilde{e} \in C \setminus e} \varphi(w(\tilde{e})) \right).$$

References

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