Generalized symmetries of Burgers equation and related algebraic structures

Dmytro R. Popovych
(Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: dmytro.popovych@imath.kiev.ua

Roman O. Popovych
(Silesian University in Opava, Czech Republic & Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine)
E-mail: rop@imath.kiev.ua

Although the Burgers equation is the most famous C-integrable model with various applications, its algebra of generalized symmetries has not been exhaustively described despite a number of relevant considerations in the literature. Filling this gap in [3], we presented a basis of this algebra in the most explicit form. We prefered to make a closed and simple proof from scratch, based on relations between the (1+1)D (linear) heat equation $\mathcal{L}_1$, the potential Burgers equation $\mathcal{L}_2$ and the Burgers equation $\mathcal{L}_3$,

$$\mathcal{L}_1\colon\ u_t=u_{xx}\quad\overset{u=\mathrm{e}^w}{\longleftrightarrow}\quad\mathcal{L}_2\colon\ w_t=w_{xx}+w_x^2\quad\overset{-2w_x=v}{\longleftrightarrow}\quad\mathcal{L}_3\colon\ v_t+vv_x=v_{xx},$$

which leads to the linearization of $\mathcal{L}_3$ to $\mathcal{L}_1$ by the Hopf–Cole transformation $v=-2u_x/u$. Another important ingredient is the exhaustive description of generalized symmetries of $\mathcal{L}_1$ in [2, Section 6]. The core of the proof is essentially simplified by using the original technique of choosing special coordinates in the associated jet space. Below, instead of the total derivative operators with respect to t and x, we use their restrictions to the solution set of the corresponding equation $\mathcal{L}_i$,

$$\mathrm{D}_x:=\partial_x+\sum_{k=0}^{\infty}z^i_{k+1}\partial_{z^i_k},\quad \mathrm{D}_t:=\partial_t+\sum_{k=0}^{\infty}\big(\mathrm{D}_x^kL^i[z^i]\big)\partial_{z^i_k},$$

where $L^1[u]:=u_{xx}$, $L^2[w]:=w_{xx}+w_x^2$, $L^3[v]:=v_{xx}-vv_x$, $z^i_0:=z^i$, the jet variable z^i_k is identified with the derivative $\partial^kz^i/\partial x^k$, $k\in\mathbb{N}$, $z^1:=u$, $z^2:=w$ and $z^3:=v$.

Recall [1, Section 6] that the algebra of generalized symmetries of the (1+1)-dimensional linear heat equation $\mathcal{L}_1$ is $\Sigma_1=\Lambda_1\Subset\Sigma_1^{-\infty}$, where $\Lambda_1:=\big\langle\mathfrak{Q}^{kl},\,k,l\in\mathbb{N}_0\big\rangle$, $\Sigma_1^{-\infty}:=\big\{\mathfrak{Z}(h)\big\}$ with $\mathfrak{Q}^{kl}:=(\mathrm{G}^k\mathrm{P}^lu)\partial_u$, $\mathrm{P}:=\mathrm{D}_x$, $\mathrm{G}:=t\mathrm{D}_x+\frac12x$, $\mathfrak{Z}(h):=h(t,x)\partial_u$, and the parameter function h runs through the solution set of $\mathcal{L}_1$. Elements of $\Sigma_1^{-\infty}$ are considered trivial generalized symmetries of $\mathcal{L}_1$ since in fact these are Lie symmetries of $\mathcal{L}_1$ that are associated with the linear superposition of solutions of $\mathcal{L}_1$. The complement subalgebra Λ_1 of $\Sigma_1^{-\infty}$ in Σ_1, which is constituted by the linear generalized symmetries of the equation $\mathcal{L}_1$, can be called the essential algebra of generalized symmetries of this equation. The algebra Λ_1 is generated by the two recursion operators P and G from the simplest linear generalized symmetry $u\partial_u$, and both the recursion operators and the seed symmetry are related to Lie symmetries.

Pulling back the elements of the algebra Σ_1 by the transformation $u=\mathrm{e}^w$, we obtain the algebra $\Sigma_2=\Lambda_2\Subset\Sigma_2^{-\infty}$ of generalized symmetries of the potential Burgers equation $\mathcal{L}_2$, which is thus isomorphic to the algebra Σ_1. As the counterparts of Λ_1 and $\Sigma_1^{-\infty}$, the subalgebra Λ_2 and the ideal $\Sigma_2^{-\infty}$ of Σ_2 are called the essential and the trivial algebras of generalized symmetries of $\mathcal{L}_2$, respectively.

Theorem 1. *The algebra of generalized symmetries of $\mathcal{L}_3$ is $\Sigma_3:=\big\langle\hat{\mathfrak{Q}}^{kl},\,(k,l)\in\mathbb{N}_0^2\setminus\{(0,0)\}\big\rangle$ with $\hat{\mathfrak{Q}}^{kl}:=\big(\mathrm{D}_x\hat{\mathrm{G}}^k\hat{\mathrm{P}}^l1\big)\partial_v$, where $\hat{\mathrm{P}}:=\mathrm{D}_x-\frac12v$ and $\hat{\mathrm{G}}:=t\mathrm{D}_x+\frac12(x-vt)$.*

Corollary 2. *The homomorphism $\varphi\colon\Lambda_2\to\Sigma_3$ of the algebra Λ_2 of essential generalized symmetries of the equation $\mathcal{L}_2$ to the entire algebra Σ_3 of generalized symmetries of the equation $\mathcal{L}_3$, which is induced by the differential substitution $-2w_x=v$, is an epimorphism, and $\ker\varphi=\langle\tilde{\mathfrak{Q}}^{00}\rangle$.*

Corollary 3. *The algebra Σ_3 of generalized symmetries of the Burgers equation $\mathcal{L}_3$ is isomorphic to the quotient algebra $A_1(\mathbb{R})^{(-)}/\langle 1\rangle$, where $A_1(\mathbb{R})^{(-)}$ is the Lie algebra associated with the first Weyl algebra $A_1(\mathbb{R})$, and $\langle 1\rangle$ is its center. Hence the algebra Σ_3 is simple and two-generated.*

The two-generation of Σ_3 as a Lie algebra means that there is a pair of its elements such that Σ_3 coincides with its subalgebra containing all successive commutators of these two elements. Examples of such pairs are in particular $\{\mathfrak{Q}^{20},\mathfrak{Q}^{03}\}$ and $\{\mathfrak{Q}^{11},\,\mathfrak{Q}^{10}-\mathfrak{Q}^{02}+\mathfrak{Q}^{30}\}$.

The commutation relations of the algebra Σ_3 are

$$[\hat{\mathfrak{Q}}^{kl},\hat{\mathfrak{Q}}^{k'l'}]=\sum_{i=1}^{\infty}i!\left(\binom{k'}{i}\binom{l}{i}-\binom{k}{i}\binom{l'}{i}\right)\hat{\mathfrak{Q}}^{k+k'-i,\,l+l'-i},$$

where $(k,l),(k',l')\in\mathbb{N}_0^2\setminus\{(0,0)\}$, and $\hat{\mathfrak{Q}}^{00}:=0$. In particular, $[\hat{\mathfrak{Q}}^{11},\hat{\mathfrak{Q}}^{kl}]=(k-l)\hat{\mathfrak{Q}}^{kl}$, i.e., the operator $\mathrm{ad}_{\hat{\mathfrak{Q}}^{11}}$ is a diagonal inner derivation in the basis $(\hat{\mathfrak{Q}}^{kl})$ and the subspace $\Gamma_m:=\langle\hat{\mathfrak{Q}}^{kl}\mid k-l=m\rangle$ of Σ_3 is the eigenspace of the operator $\mathrm{ad}_{\hat{\mathfrak{Q}}^{11}}$ corresponding to the eigenvalue m. The Jacobi identity for the Lie bracket implies that $[\Gamma_m,\Gamma_{m'}]\subseteq\Gamma_{m+m'}$ for any $m,m'\in\mathbb{Z}$. As a result, the decomposition of the algebra Σ_3 as the direct sum of its subspaces Γ_m, $\Sigma_3=\oplus_{m\in\mathbb{Z}}\Gamma_m$, is a $\mathbb{Z}$-grading of this algebra.

Corollary 4. *The space of generalized symmetries of the Burgers equation $\mathcal{L}_3$ that are of order not greater than n is $\Sigma_3^n:=\big\langle\hat{\mathfrak{Q}}^{kl},\,(k,l)\in\mathbb{N}_0^2\setminus\{(0,0)\},\,k+l\leqslant n\big\rangle$, and $\dim\Sigma_3^n=\frac12 n(n+3)$.*

Recall that the maximal Lie invariance algebra $\mathfrak{g}^{\rm B}$ of $\mathcal{L}_3$ is five-dimensional, $\mathfrak{g}^{\rm B}=\langle\hat{\mathcal{P}}^t,\hat{\mathcal{D}},\hat{\mathcal{K}},\hat{\mathcal{P}}^x,\hat{\mathcal{G}}^x\rangle$, where $\hat{\mathcal{P}}^t=\partial_t$, $\hat{\mathcal{D}}=2t\partial_t+x\partial_x-v\partial_v$, $\hat{\mathcal{K}}=t^2\partial_t+tx\partial_x+(x-tv)\partial_v$, $\hat{\mathcal{G}}^x=t\partial_x+\partial_v$, $\hat{\mathcal{P}}^x=\partial_x$. In fact, the spaces $\Sigma_3^1=\big\langle\hat{\mathfrak{Q}}^{01},\,\hat{\mathfrak{Q}}^{10}\big\rangle$ and $\Sigma_3^2=\big\langle\hat{\mathfrak{Q}}^{01},\,\hat{\mathfrak{Q}}^{10},\,\hat{\mathfrak{Q}}^{02},\,\hat{\mathfrak{Q}}^{11},\,\hat{\mathfrak{Q}}^{20}\big\rangle$ of generalized symmetries of the Burgers equation $\mathcal{L}_3$ that are of order not greater than one and two are closed with respect to Lie bracket of generalized vector fields, i.e., they are a one- and a five-dimensional subalgebras of Σ_3, respectively. They are constituted by the canonical evolution forms of elements of the (nil)radical $\langle\hat{\mathcal{P}}^x,\hat{\mathcal{G}}^x\rangle$ of $\mathfrak{g}^{\rm B}$ and of the entire algebra $\mathfrak{g}^{\rm B}$ and thus respectively isomorphic to these algebras. More specifically, the basis elements $\hat{\mathcal{P}}^t$, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{G}}^x$ and $\hat{\mathcal{P}}^x$ of $\mathfrak{g}^{\rm B}$ are associated, up to their signs, with the elements $2\hat{\mathfrak{Q}}^{02}$, $4\hat{\mathfrak{Q}}^{11}$, $2\hat{\mathfrak{Q}}^{20}$, $2\hat{\mathfrak{Q}}^{10}$ and $2\hat{\mathfrak{Q}}^{01}$ of Σ_3^2, respectively.

The algebra Σ_3^1 is the only finite-dimensional maximal Abelian subalgebra of Σ_3. We conjecture that the algebra Σ_3^2 is the only finite-dimensional maximal subalgebra of Σ_3.

Using the results of [1], we can describe maximal Abelian subalgebras of Σ_3. Each of the other maximal Abelian subalgebras of Σ_3 is infinite-dimensional since it contains a subalgebra of the form $\langle(\mathrm{D}_x\mathrm{Q}^k1)\partial_u,\,k\in\mathbb{N}\rangle$ with a nonconstant polynomial Q of $\hat{\mathrm{G}}$ and $\hat{\mathrm{P}}$. Moreover, it coincides with the centralizer of an element of $\Sigma_3\setminus\langle\hat{\mathfrak{Q}}^{01},\hat{\mathfrak{Q}}^{10}\rangle$, which necessarily belongs to it, or, equivalently, with the centralizer of any element of its relative complement to $\langle\hat{\mathfrak{Q}}^{01},\hat{\mathfrak{Q}}^{10}\rangle$.

We also show that the two well-known recursion operators of the Burgers equation and its two seed generalized symmetries, which are evolution forms of its Lie symmetries, suffice to generate this algebra within the framework of the formal approach, whereas the zero generalized symmetry is sufficient as the only seed symmetry if the recursion operators are interpreted as Bäcklund transformations for the corresponding tangent bundle.

References

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