Topological, metric and fractal analysis of infinite Bernoulli convolutions governed by convergent positive series

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An *infinite Bernoulli convolution* governed by an absolutely convergent series $r_0 = \sum\limits_{n=1}^{\infty} u_n$ is defined as the distribution of the random variable $\zeta = \sum\limits_{n=1}^{\infty} \zeta_n u_n$, where (ζ_n) is a sequence of independent random variables taking values 0 and 1 with probabilities p_0 and $p_1 = 1 - p_0$, respectively.

A *generalized infinite Bernoulli convolution* is defined as the distribution of the random variable $\xi = \sum\limits_{n=1}^{\infty} \xi_n u_n$, where (ξ_n) is a sequence of independent random variables taking values in $\{0, 1, ..., r\}$ with respective probabilities $p_0, p_1, ..., p_r$. The main object of study in this report is the distribution of the random variable

$$\xi = \sum_{n=1}^{\infty} \frac{\xi_n}{s^n} = \Delta^{r_s}_{\xi_1 \xi_2 \ldots \xi_n \ldots},$$

where s and r are natural parameters such that $1 < s \leq r$.

The set E_ξ of values of the random variable ξ is the segment $[0, \frac{r}{s-1}]$.

According to the Jessen–Wintner theorem, the distribution of a random variable is either purely discrete, purely singular, or purely absolutely continuous. We are particularly interested in the conditions under which the distribution is concentrated on a set of Lebesgue measure zero and in determining the fractal dimension of such a set.

The main difficulties in obtaining a complete answer to these questions arise from the ambiguity of number representations in the numeral system with base s and the redundant alphabet $A = \{0, 1, ..., r\}$.

The report is devoted to the case $s = 3 = r$. From here on, we write

$$\Delta^{r_s}_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots} = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots} = \sum_{n=1}^{\infty} 3^{-n} \alpha_n, \quad \alpha_n \in A \equiv \{0, 1, 2, 3\}.$$

Lemma 1. *The set $C[\Delta; \{0, 1, 3\}] = \{x : \ x = \Delta_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots}, \alpha_n \in \{0, 1, 3\}\}$ is a nowhere dense, N-self-similar set of zero Lebesgue measure, whose Hausdorff dimension is equal to $\log_3 \frac{3+\sqrt{5}}{2}$.*

Theorem 2. *Let $p_0 p_1 p_2 p_3 = 0$.*

1. *If $p_i = p_{i+1} = p_{i+2} = \frac{1}{3}$, then ξ has a* uniform distribution *on the unit interval.*
2. *If there exists p_j such that $0 \neq p_j \neq \frac{1}{3}$, then the distribution of the random variable ξ s* singular*, and:*

   2.1) if two of the probabilities are zero, then ξ has a Cantor-type distribution *with spectrum of fractal dimension $\log_3 2$;*

   2.2) if exactly one is zero and $p_1 p_2 = 0$, then ξ has a Cantor-type distribution *with spectrum of fractal dimension $\log_3 \frac{3+\sqrt{5}}{2}$;*

2.3) if exactly one is zero and $p_0p_3 = 0$, then ξ has a singular distribution with a strictly increasing distribution function, *and the fractal dimension of the essential support of its density is $-\log_3 p_1^{p_1} p_2^{p_2} p_i^{p_i}$.*

Theorem 3. *If $p_1 = \frac{1}{3} = p_2$, then the random variable $\xi = \Delta_{\xi_1\xi_2\ldots\xi_n\ldots}$ with independent digits $\xi_n \in \{0, 1, 2, 3\}$ having probabilities p_0, p_1, p_2, p_3 has an absolutely continuous distribution. Moreover, the distribution of ξ is the convolution of the uniform distribution on $[0; 1]$ and a singular Cantor-type distribution. In all other cases, the distribution of ξ is singular.*

Corollary 4. *Every infinite Bernoulli convolution governed by the series*

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3^2} + \ldots + \frac{1}{3^n} + \frac{1}{3^n} + \frac{1}{3^n} + \ldots,$$

has a purely singular distribution.

Lemma 5. *The sum of two independent singularly distributed random variables*

$$\theta = \sum_{n=1}^{\infty} \theta_n 3^{-n} \text{ and } \varepsilon = \sum_{n=1}^{\infty} \varepsilon_n 3^{-n},$$

where (θ_n) and (ε_n) are sequences of independent random variables taking values in $\{0, 2\}$ and $\{0, 1\}$, respectively, with probabilities $u, 1 - u$ and $v, 1 - v$, has a singular distribution whose spectrum is the interval $[0; \frac{3}{2}]$.

Theorem 6. *If $p_0 = (p_0 + p_1)(p_0 + p_2)$ then the distribution of ξ is the convolution of two Cantor-type distributions, namely, the distributions of the random variables*

$$\theta = \Delta_{\theta_1\theta_2\ldots\theta_n\ldots}, \quad \varepsilon = \Delta_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots},$$

where (θ_n) and (ε_n) are sequences of independent random variables taking values in $\{0, 2\}$ and $\{0, 1\}$, with probabilities $p_0 + p_1$ and $1 - (p_0 + p_1)$, and $p_0 + p_2$ and $1 - (p_0 + p_2)$, respectively.

Remark 7. The proof of Theorem 3 is based on the method of characteristic functions (integral transforms) and the method of extracting the absolutely continuous component.

Remark 8. The problem of describing the topological, metric, and fractal properties of the essential support of the density

$$N_\xi = \{x : F'_\xi(x) > 0 \text{ або } F'_\xi(x) \text{ does not exist}\}$$

of the distribution ξ under the condition $p_0p_1p_2p_3 \neq 0$, remains open.

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