A Banach space characterization of (sequentially) Ascoli spaces

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The talk is based on my recent article [4].

One of the basic theorems in Analysis is the Ascoli theorem which states that if X is a k-space, then every compact subset of $C_k(X)$ is evenly continuous, see Theorem 3.4.20 in [2]. Being motivated by the Ascoli theorem we introduced and studied in [1] the class of Ascoli spaces. A Tychonoff space X is called an *Ascoli space* if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, that is the map $X \times \mathcal{K} \ni (x, f) \mapsto f(x)$ is continuous. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [5, p.45].

Being motivated by the classical notion of c_0-barrelled locally convex spaces we defined in [3] a Tychonoff space X to be *sequentially Ascoli* if every convergent sequence in $C_k(X)$ is equicontinuous. Clearly, every Ascoli space is sequentially Ascoli, but the converse is not true in general.

Let X be a Tychonoff space, and let E'_β be the dual space of a locally convex space E. We shall say that a map $T : X \to E'$ is *almost k-compact* (resp., *almost k-sequential*) if it is weak* continuous and there are a neighborhood U of zero in E and a compact subset (resp., a null sequence) K of $C_k(X)$ such that the family $\{T_E(x, a) : \ a \in U\}$ is contained in the absolutely convex closed hull $\overline{\mathrm{acx}}(K)$ of K. Now we formulate the main result of the talk.

Theorem 1. *For a Tychonoff space X, the following assertions are equivalent:*

(i) *X is an Ascoli (resp., sequentially Ascoli) space;*
(ii) *for each cardinal Γ, every k-continuous and almost k-compact (resp., almost k-sequential) map $T : X \to \ell_\infty(\Gamma)$ is continuous;*
(iii) *for each Banach space E, every k-continuous and almost k-compact (resp., almost k-sequential) map $T : X \to E'_\beta$ is continuous.*

References

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