Liminal $\mathrm{SL}_2\mathbb{Z}_p$-representations and odd-th cyclic covers of genus one two-bridge knots

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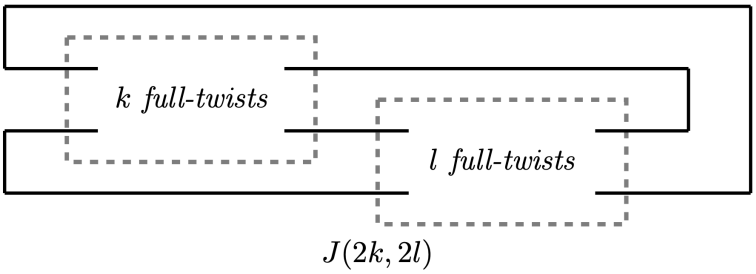
We briefly survey a joint work with Ryoto Tange and Jun Ueki [STU25].

Theorem 1. *It is known that every genus one two-bridge knot is realized as a* double twist knot *of type $J(2k, 2l)$ with $(0, 0) \neq (k, l) \in \mathbb{Z}^2$ defined by the following diagram.*



$J(2k, 2l)$

Definition 2. Let p be a prime number. When n ranges over natural numbers, the rings $\mathbb{Z}/p^n\mathbb{Z}$ naturally form an inverse system. The inverse limit of this system is called the ring of p-adic integers and denoted by $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Definition 3. Let π be a group.
(1)A function $\chi : \pi \to \mathbb{Z}_p$ is called an $\mathrm{SL}_2\mathbb{Z}_p$-*character* if there exists an SL_2-representation ρ over an extension of $\mathbb{Z}_p$ such that $\chi = \operatorname{tr} \rho$ holds.
(2)An $\mathrm{SL}_2\mathbb{Z}_p$-character is said to be *liminal* if it is absolutely reducible and its every open neighborhood contains an absolutely irreducible $\mathrm{SL}_2\mathbb{Z}_p$-character. Here, a neighborhood refers to a neighborhood with respect to the p-adic distance on the character variety.

Theorem 4. *Let $K = J(2k, 2l)$ be a genus one two-bridge knot in S^3. If a prime number p divides the size of the 1st homology group of some odd-th cyclic branched cover of K, then its group $\pi_1(S^3 - K)$ admits a liminal* $\mathrm{SL}_2\mathbb{Z}_p$*-character.*

In the proof of this theorem, a nature of certain Lucas-type sequences plays a key role.

Example 5. The sequence (L_n) starting with $L_0 = 2$, $L_1 = 1$, and defined by $L_n = L_{n-1} + L_{n-2}$ is called the Lucas sequence. Calculating from the smallest terms, we get:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \textit{etc.}$$

Focusing on the prime factors of the odd-indexed terms, we get:

$$L_3 = 2^2, L_5 = 11, L_7 = 29, L_9 = 2^2 \times 19, L_{11} = 199, L_{13} = 521, L_{15} = 11 \times 31, \textit{etc.}$$

The prime factors are either 2, or have a last digit of 1 or 9. This observation can be generalized as follows: If a prime number p divides L_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then $p = 2$ or the Legendre symbol satisfies $\left(\dfrac{5}{p}\right) = 1$.

Proposition 6. *Let $m \in \mathbb{Z}$. Let a and b be the solutions of the equation $t^2 - t + m = 0$, and define $L_n = a^n + b^n$ for any $n \in \mathbb{N}$. If a prime number p divides L_{2n+1} for some $n \in \mathbb{Z}_{\geq 0}$, then the Legendre*

symbol satisfles

$$\left(\frac{4m^2-m}{p}\right)=1.$$

Proof. Define $F_n=\dfrac{a^n-b^n}{a-b}$ for any $n\in\mathbb{N}$. Then, $F_n\in\mathbb{Z}$, and

$$L_n^2+(4m-1)F_n^2=4m^n$$

holds. If p divides L_{2n+1} for some $n\in\mathbb{Z}_{\geq 0}$, then

$$(4m-1)F_{2n+1}^2\equiv 4m^{2n+1}\bmod p.$$

So, $m(4m-1)$ is a square mod p. □

Proof of Theorem 4 (Outline). A Seifert matrix of $J(2k,2l)$ is given by $V=\begin{pmatrix}k & 1\\ 0 & l\end{pmatrix}$, so the Alexander polynomial is

$$\Delta_{J(2k,2l)}(t)=\det\left(tV-V^{\perp}\right)=klt^2+(1-2kl)t+kl.$$

Let α and β denote the solutions $\Delta_K(t)=mt^2-(1-2m)t+m=0$, where $m=kl$.
Then, by using Fox–Weber formula, we can write

$$r_{2n+1}=|\mathrm{Res}(t^n-1,\Delta_K(t))|=m^{2n+1}(2-\alpha^{2n+1}-\beta^{2n+1}).$$

Let a and b denote the solutions of $t^2-t+m=0$. Then, $\{a^2,b^2\}=\{-m\alpha,-m\beta\}$, and we obtain $r_{2n+1}=L_{2n+1}^2$. Therefore, by **Proposition 6**, if a prime number p divides r_{2n+1} for some $n\in\mathbb{Z}_{\geq 0}$, then the Legendre symbol satisfies $\left(\dfrac{4k^2l^2-kl}{p}\right)=1$.

On the other hand, let $S_*(z)\in\mathbb{Z}[z]$ denotes the Chebyshev polynomial of the 2nd kind and define

$$f_{k,l}(x,y)=S_l(z)-(1+(-x^2+y+2)S_{k-1}(y)(S_k(y)-S_{k-1}(y))S_{l-1}(z),$$

$$z=2+(y-2)(-x^2+y+2)S_{m-1}^2(y).$$

Then Tran's calculation [Tra18] and Hensel's lemma assure that liminal $\mathrm{SL}_2\mathbb{Z}_p$ characters correpsonds to intersection points $(\pm\sqrt{4-\dfrac{1}{kl}},-\dfrac{1}{kl})$ of the curves $f_{k,l}(x,y)=0$ and $y-2=0$ in $\mathbb{Z}_p^2$. This completes the proof. □

Remark 7. The analogies between knots and primes, or 3-manifolds and number rings have played important roles since the era of Gauss (cf.[Mor24]). In modern times, among other things, the analogy between the Alexander–Fox theory of $\mathbb{Z}$-covers and the Iwasawa theory of $\mathbb{Z}_p$-extensions of number fields, and that between deformation theories of knot group representations (e.g., Thurston's hyperbolic deformation) and Galois representations (e.g., due to Hida–Mazur) have been pointed out. There are special interests in irreducible $\mathrm{SL}_2\mathbb{Z}_\mathrm{p}$-representations whose residual representations are reducible. In our study [STU25], following Mazur [Maz11], we aimed to "go the other way".

References

[Maz11] Barry Mazur, "*How can we construct abelian Galois extensions of basic number fields?*", Bull. Amer. Math. Soc. (N.S.) **48** (2011), no. 2, 155–209. MR 2774089

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