On Koebe-Bloch theorem for mappings with inverse Poletsky inequality

Evgeny Sevost'yanov
(Zhytomyr Ivan Franko State University; Institute of Applied Mathematics and Mechanics, Slov'yans'k)
E-mail: esevostyanov2009@gmail.com

Valery Targonskii
(Zhytomyr Ivan Franko State University)
E-mail: w.targonsk@gmail.com

Let us recall the formulation of the classical Koebe theorem.

Theorem A. *Let $f:\mathbb{D}\rightarrow \mathbb{C}$ be an univalent analytic function such that $f(0)=0$ and $f^{\,\prime}(0)=1$. Then the image of f covers the open disk centered at 0 of radius one-quarter, that is, $f(\mathbb{D})\supset B(0, 1/4)$.*

The main fact contained in the paper is the statement that something similar has been done for a much more general class of spatial mappings. Below $dm(x)$ denotes the element of the Lebesgue measure in ${\Bbb R}^n$. Everywhere further the boundary ∂A of the set A and the closure $\overline{A}$ should be understood in the sense of the extended Euclidean space $\overline{{\Bbb R}^n}$. Recall that, a Borel function $\rho:{\Bbb R}^n\,\rightarrow [0,\infty]$ is called *admissible* for the family Γ of paths γ in ${\Bbb R}^n$, if the relation

$$\int\limits_{\gamma}\rho (x)\,|dx|\geqslant 1 \tag{1}$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in {\rm adm} \,\Gamma$. The *modulus* of Γ is defined by the equality

$$M(\Gamma)=\inf\limits_{\rho \in \,{\rm adm}\,\Gamma} \int\limits_{{\Bbb R}^n} \rho ^n (x)\,dm(x)\,. \tag{2}$$

Let $y_0\in {\Bbb R}^n,$ $0<r_1<r_2<\infty$ and

$$A=A(y_0, r_1,r_2)=\left\{ y\,\in\,{\Bbb R}^n: r_1<|y-y_0|<r_2\right\}\,. \tag{3}$$

Given $x_0\in{\Bbb R}^n,$ we put $B(x_0, r)=\{x\in {\Bbb R}^n: |x-x_0|< r\}\,,$ ${\Bbb B}^n=B(0, 1)\,,$ $S(x_0,r) = \{ x\,\in\,{\Bbb R}^n : |x-x_0|=r\}\,.$ A mapping $f:D\rightarrow {\Bbb R}^n$ is called *discrete* if the pre-image $\{f^{-1}\left(y\right)\}$ of any point $y\,\in\,{\Bbb R}^n$ consists of isolated points, and *open* if the image of any open set $U\subset D$ is an open set in ${\Bbb R}^n.$ Given sets $E,$ $F\subset\overline{{\Bbb R}^n}$ and a domain $D\subset {\Bbb R}^n$ we denote by $\Gamma(E,F,D)$ the family of all paths $\gamma:[a,b]\rightarrow \overline{{\Bbb R}^n}$ such that $\gamma(a)\in E,\gamma(b)\in\,F$ and $\gamma(t)\in D$ for $t \in (a, b).$ Given a mapping $f:D\rightarrow {\Bbb R}^n,$ a point $y_0\in \overline{f(D)}\setminus\{\infty\},$ and $0<r_1<r_2<r_0=\sup\limits_{y\in f(D)}|y-y_0|,$ we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma)\in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0,r_1,r_2)).$ Let $Q:{\Bbb R}^n\rightarrow [0, \infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0\in \overline{f(D)}\setminus\{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2))\leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y)\cdot \eta^n(|y-y_0|)\, dm(y) \tag{4}$$

holds for any Lebesgue measurable function $\eta: (r_1,r_2)\rightarrow [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2}\eta(r)\, dr\geqslant 1\,. \tag{5}$$

The relations (4) are proved for different classes of mappings, see e.g. [2].

Set $q_{y_0}(r)=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{S(y_0,r)}Q(y)\,d\mathcal{H}^{n-1}(y)$, where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi: D\rightarrow\mathbb{R}$ has a *finite mean oscillation* at a point $x_0\in D$, write $\varphi\in FMO(x_0)$, if $\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)}|\varphi(x)-\overline{\varphi}_{\varepsilon}|\ dm(x)<\infty$, where $\overline{\varphi}_{\varepsilon}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\,\varepsilon)}\varphi(x)\,dm(x)$ and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We also say that a function $\varphi: D\rightarrow\mathbb{R}$ has a finite mean oscillation at $A\subset\overline{D}$, write $\varphi\in FMO(A)$, if φ has a finite mean oscillation at any point $x_0\in A$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2}\sqrt{1+{|y|}^2}}\,,\qquad x\ne\infty\ne y\,,$$

and let $h(E):=\sup\limits_{x,y\in E}h(x,y)$ be a chordal diameter of a set $E\subset\overline{\mathbb{R}^n}$ (see, e.g., [1, Definition 12.1]).

Given a continuum $E\subset D$, $\delta>0$ and a Lebesgue measurable function $Q:\mathbb{R}^n\rightarrow[0,\infty]$ we denote by $\mathfrak{F}_{E,\delta}(D)$ the family of all open discrete mappings $f:D\rightarrow\mathbb{R}^n$, $n\geqslant 2$, satisfying relations (4)–(5) at any point $y_0\in\overline{\mathbb{R}^n}$ such that $h(f(E))\geqslant\delta$. The following statement holds, cf. [3].

Theorem 1. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $B(x_0,\varepsilon_1)\subset D$ for some $\varepsilon_1>0$.*

Assume that, $Q\in L^1(\mathbb{R}^n)$ and, in addition, one of the following conditions hold:

1) $Q\in FMO(\overline{\mathbb{R}^n})$;

2) for any $y_0\in\overline{\mathbb{R}^n}$ there is $\delta(y_0)>0$ such that

$$\int\limits_{0}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}=\infty\,. \tag{6}$$

Then there is $r_0>0$, which does not depend on f, such that

$$f(B(x_0,\varepsilon_1))\supset B_h(f(x_0),r_0)\qquad\forall\ f\in\mathfrak{F}_{E,\delta}(D)\,,$$

where $B_h(f(x_0),r_0)=\{w\in\overline{\mathbb{R}^n}: h(w,f(x_0))<r_0\}$.

Remark 2. The condition $Q\in FMO(\infty)$ of the condition (6) for $y_0=\infty$ must be understood as follows: these conditions hold for $y_0=\infty$ if and only if the function $\widetilde{Q}:=Q\left(\frac{y}{|y|^2}\right)$ satisfies similar conditions at the origin.

The result mentioned above is obtained in [4].

References

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