Normal subgroups of iterated wreath products of symmetric groups

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In this research we continue our previous investigation of wreath product normal structure [1, 2]. Normal subgroups and there structures for finite and infinite iterated wreath products $S_{n_1} \wr \ldots \wr S_{n_m}$, $n, m \in \mathbb{N}$ and $A_n \wr S_n$ are founded.

Let $k(\pi)$ be the number of cycles in decomposition of permutation π of degree n. The number $n - k(\pi)$ is denoted by $dec(\pi)$, and is called a decrement [6] of permutation π. If $\pi_1, \pi_2 \in S_n$, then the following formula holds:

$$dec(\pi_1 \cdot \pi_2) = dec(\pi_1) + dec(\pi_2) - 2m, m \in \mathbf{N}, \quad (1)$$

Definition 1. The permutational *subwreath product* $G \wr\wr H$ is the semi-direct product $G \ltimes \tilde{H}^X$, where G acts on the subdirect product [4] $\tilde{H}^X$ by the respective permutations of the subdirect factors. Provided the specification of $\tilde{H}^X$ is established separately.

Definition 2. The set of elements from $S_n \wr S_n, n \geqslant 3$ which presented by the tableaux of form: $[e]_0,\ [a_1, a_2, \ldots, a_n]_1$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_1) = 2k, k \in \mathbb{N}, \quad (2)$$

be called a generalized alternating group of first level $\widetilde{A}_n^{(1)}$, and denote this set by $E \wr\wr \widetilde{A}_n$. Note that condition (2) uniquely identifies subdirect product.

We spread this definition on 3-multiple wreath product by recursive way.

Definition 3. *The subgroup* $E \wr \widetilde{A}_n^{(1)}$ *be denoted by* $\widetilde{A}_n^{(2)}$.

Theorem 4. *The subgroup* $\widetilde{A}_n^{(1)}$ *has* ***normal rank*** $n-1$ [7] *in* $S_n \wr S_n,\ n \geqslant 3$ *provided* $n \equiv 1 (mod 2)$ *and* ***normal rank*** n *iff* $n \equiv 0 (mod 2)$ *and* $n \geqslant 3$.

Theorem 5. *The subgroup* $\widetilde{A}_3^{(1)}$ *of* $S_3 \wr S_3$ *has the structure* $\widetilde{A}_3^{(1)} \simeq (C_3 \times C_3 \times C_3) \rtimes (C_2 \times C_2)$.

The structure of subgroup $\widetilde{A}_n^{(1)} \leq S_n \wr S_n$ *is* $\widetilde{A}_n^{(1)} \simeq (\prod_{i=1}^{n} A_n) \rtimes (\prod_{i=1}^{n-1} C_2)$.

Definition 6. The set of elements from $S_n \wr S_n \wr S_n, n \geqslant 3$ presented by the tables [3] form:
$[e]_0,\ [e, e, \ldots, e]_1, [a_1, a_2, \ldots, a_n]_2$, satisfying the following condition

$$\sum_{i=1}^{n} dec([a_i]_2) = 2k, k \in \mathbb{N}, \quad (3)$$

be denoted by $\widetilde{A}_{n^2}^{(2)}$. Note that condition (3) uniquely identifies subdirect product in $\prod_{i=1}^{n^2} S_n$ as base of subwreath product, the similar subdirect product describing commutator of wreath product was investigated by us in [9] in research of pronormality it appears in [8].

Proposition 7. *The subgroup* $\widetilde{A}_n^{(1)} \lhd S_n \wr S_n$ *as well as* $\widetilde{A}_n^{(2)} \lhd S_n \wr S_n \wr S_n$. *Furthermore* $\widetilde{A}_n^{(2)} \lhd \widetilde{A}_{n^2}^{(2)}$.

Definition 8. *A subgroup in* $S_n \wr S_n$ *is called* $\widetilde{T_n}$ *if it consists of:*

(1) elements of $E \wr A_n$ *,*

(2) elements with the tableau [3] *presentation* $[e]_1,\ [\pi_1,\ldots,\pi_n]_2$*, that* $\pi_i \in S_n \setminus A_n$.

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure

$$\tilde{T}_n \simeq (\underbrace{A_n \times A_n \times \cdots \times A_n}_{n}) \rtimes C_2 \simeq \underbrace{S_n \boxplus S_n \ldots \boxplus S_n}_{n},$$

where the operation $\boxplus$ *of a subdirect product is subject of item 1) and 2)*

Remark 9. The order of $\widetilde{T_n}$ is $\frac{(n!)^n}{2^{n-1}}$.

Definition 10. The unique minimal normal subgroup is called the monolith.

Theorem 11. *The monolith of* $S_n \wr S_m$ *is* $e \wr A_m$.

Definition 12. The set of elements from $\mathop{\wr}\limits_{i=1}^{k} S_{n_i}, n_i \geqslant 3$ with depth m

satisfying the following condition

$$\sum_{i=1}^{n^j} dec([a_i]_j) = 2t, t \in \mathbb{N},\ m \leq j \leq k,\quad [a_i]_j = e,\ whenever\, j = \overline{1, m-1} \tag{4}$$

be called $\widetilde{A}_{n^j}^{(m,k)}$, where $m < k$.

Theorem 13. *The order of normal subgroup* $\widetilde{A}_{n^j}^{(1,k)}$ *is* $(\frac{1}{2})^k \cdot (n!)^{(\frac{n^{(k+1)}-1}{n-1})}$ *and the order of the quotient* $\mathop{\wr}\limits_{i=1}^{k} S_{n_i} \Big/ \widetilde{A}_{n^j}^{(1,k)}$ *is* 2^k*. The order of generalized alternating group of* k*-th level* $\widetilde{A}_{n^k}^{(k)}$ *is* $2^{n^k - 1}$.

Theorem 14. *Proper normal subgroups in* $S_n \wr S_m$*, where* $n, m \geq 3$ *with* $n, m \neq 4$ *are of the following types:*

(1) subgroups that act only on the second level are

$$\tilde{A}_m^{(1)},\ \widetilde{T_m},\ E \wr S_m, E \wr A_m,$$

(2) subgroups that act on both levels are $A_n \wr\wr \widetilde{A}_m^{(1)},\ S_n \wr\wr \widetilde{B}_m^{(1)},\ A_n \wr S_m$,

wherein the subgroup $S_n \wr\wr \widetilde{A_m} \simeq S_n \curlywedge (\underbrace{S_m \boxtimes S_m \boxtimes S_m \ldots \boxtimes S_m}_{n})$ *endowed with the subdirect product satisfying to condition* (2).

The group $\tilde{B}_n^{(1)}$ is isomorphic copy of $\tilde{A}_n^{(1)}$ which is realized by another embedding in $AutX^{[2]}$. The lattice of invariant subgroups for $S_n \wr S_n$, $n \equiv 0\,(\mathrm{mod} 2)$ is presented on Fig. 1.

Lemma 15. *Any 2 normal subgroups* $N_l, N_j \lhd \mathop{\wr}\limits_{i=1}^{k} S_{n_i}, n_i \geqslant 3$ *are mutually commutative* $N_l N_j = N_j N_l$.

A group N of $AutX^{[k]}$ is said to be a group of depth $d = d\,(N)$ if N contain trivial permutations on levels $1, ..., d-1$, and first non-trivial permutation on level number $d \leq k$.

A group N of $AutX^{[k]}$ is said to be a group of height $h(N)$, where k is multiplicity of wreath product, if the difference $h = k - d\,(N)$, where $d\,(N)$ is depth of N. The set of normal subgroups of height h in $AutX^{[k]}$ is denoted by $N\,(h, k)$. Let us denote the number of normal subgroups of height h in $AutX^{[k]}$ as $n\,(h, k)$. We denote the i-th normal subgroup of height h in $AutX^{[k]}$ as $N_i\,(h, k)$. According to Theorem 14 $N(2, 2) = \{N_i(2, 2) : 1 \leq i \leq 5\}$.

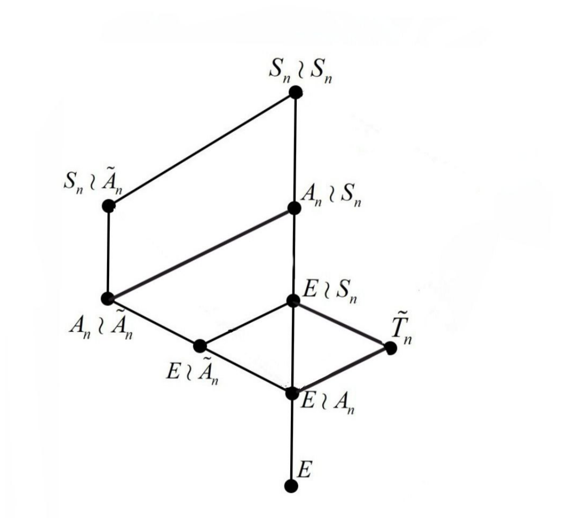

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FIGURE 14.1. Lattice of invariant subgroups $S_n \wr S_n$ for the case $n \equiv 0 \,(\mathrm{mod} 2)$

Theorem 16. *The full list of normal subgroups of $W = S_n \wr S_n \wr S_n \simeq AutX^{[3]}$ consists of 50 normal subgroups.*

1 *subgroups of height 2 on base of set $N(2,2)$ takes form $E \wr N_i(2,2)$: $E \wr A_n \wr\wr \widetilde{A}_n^{(1)}, E \wr A_n \wr S_n$ $E \wr S_n \wr\wr \widetilde{A}_n^{(1)}$, $E \wr S_n \wr\wr \widetilde{A}_n^{(1)}$ $E \wr S_n \wr\wr \widetilde{B_n^{(1)}}$, $E \wr S_n \wr S_n$. There are 5 new subgroups.*

2 ***subgroups of height 2 in*** *$AutX^{[3]}$ that based on new subgroups of X^3: $\tilde{A}_{n^2}$ or $\tilde{B}_{n^2}$: $E \wr S_n \wr\wr \tilde{A}_{n^2}$, $E \wr A_n \wr\wr \tilde{B}_{n^2}$, $E \wr A_n \wr\wr \tilde{A}_{n^2}$, $E \wr A_n \wr\wr \tilde{B}_{n^2}$, $E \wr \tilde{A}_n^{(2)} \wr\wr \tilde{A}_{n^2}^{(2)}$, and subclass with subgroup H_i on X^1 such that $H_i \in \{\tilde{A}_n^{(1)}, \tilde{B}_n^{(1)}, \tilde{T}_n^{(1)}\}$ so subgroups takes form $E \wr\wr H_i \wr\wr \tilde{A}_{n^2}$, $E \wr\wr H_i \wr\wr \tilde{B}_{n^2}$. Therefore this class has 10 new subgroups.*

3 *subgroups of $N(2,3)$ having a level subgroups on $X^2 \subset X^{[3]}$ such that from last level of $N(2,2)$ and one of them on X^3 There are 3 subgroup such level subgroups $H_i \in \{\tilde{A}_n, \tilde{B}_n, S_n\}$. Thus, there are 9 subgroups of form: $E \wr \widetilde{A}_n \wr H_i, E \wr \widetilde{T}_n \wr H_i$, $E \wr \widetilde{B}_n \wr H_i$.*

Thus, the total number of normal subgroups in of height 2 is 24.

4 *subgroups of height 1 based on normal subgroups of type $N(1,2)$: $\prod_{i=1}^{9} A_n$, $\widetilde{T}_n$, $\widetilde{A}_n$, $\widetilde{B}_n$, $\prod_{i=1}^{9} S_n$. And new subgroups of type $N(1,3)$ $\widetilde{A}_{n^2}$, $\widetilde{B}_{n^2}$, $\widetilde{T_n^{(2)}}$. Hence, here are 8 new subgroups.*

5 *subgroups of height 3 admit on first level S_n or A_n, on second one of $\{\tilde{A}_n, \tilde{B}_n, S_n^3\}$, on third $\{\tilde{A}_{n^2}, \tilde{B}_{n^2}, S_n^9\}$. Thus, there 18 normal subgroups in $N(3,3)$.*

Remark 17. Note that $E \wr\wr \widetilde{A}_n^{(1)} \simeq \widetilde{A}_n^{(2)}$ contains in the family $E \wr N_i(S_n \wr S_n)$.

We denote by $Aut_f X^*$ the group of all finite automorphism of spherically homogeneous rooted tree.

Theorem 18. *Let $H \lhd Aut_f X^*$ having depth k, then H contains k-th level subgroup P having all even vertex permutations $p_{ki} \in A_n$ on X^k and trivial permutations in vertices of rest of levels.*

Furthermore P is normal in $Aut_f X^*$ provided k is last active level of $Aut_f X^*$.

Theorem 19. *The order of normal subgroup* $\widetilde{A}_{n^j}^{(1,k)}$ *is* $(\frac{1}{2})^k \cdot (n!)^{(\frac{n^{(k+1)}-1}{n-1})}$ *and the order of the quotient* $\overset{k}{\underset{i=1}{\wr}} S_{n_i} \Big/ \widetilde{A}_{n^j}^{(1,k)}$ *is* 2^k. *The order of generalized alternating group of k-th level* $\widetilde{A}_{n^k}^{(k)}$ *is* 2^{n^k-1}.

To study the parity of elements at all levels, we factorize by the *maximal normal subgroup* $\tilde{A}_{n^i}^{1,k}$ that contains the *generalized alternating group* of permutations at each level.

Lemma 20. *The following homomorphism* $W_n/\tilde{A}_{n^k}^k \cong W_{n-1} \wr C_2$ *holds.*

Theorem 21. *The quotient* $\overset{k}{\underset{i=1}{\wr}} S_{n_i}$ *by* $\widetilde{A}_{n^j}^{(1,k)}$ *is the following group* $\prod_{i=1}^{k} \mathbb{Z}_2$.

References

[1] *Skuratovskii R.V.* Invariant structures of wreath product of symmetric groups. Naukovuy Chasopus of Science hour writing of the National Pedagogical University named after M.P. Dragomanova. (in ukrainian) Series 01. Physics and Mathematics. — 2009. Issue 10. — P. 163 – 178.

[2] *Ruslan Skuratovskii.* Normal subgroups of iterated wreath products of symmetric groups and alternating with symmetric groups. 2022, Source: [https://doi.org/10.48550/arXiv.2108.03752]

[3] Kaloujnine L. A. Sur les p-group de Sylow. // *C. R. Acad. Sci. Paris.* — 1945. — **221**. — P. 222–224.

[4] *Birkhoff, Garrett* (1944), "Subdirect unions in universal algebra", *Bulletin of the American Mathematical Society*, 50 (10): 764-768, doi:10.1090/S0002-9904-1944-08235-9, ISSN 0002-9904, MR 0010542.

[5] Drozd Y.A., Skuratovskii R.V. Generators and relations for wreath products. *Ukr. math. journ.* - 60, n. 7. - 2008.- S. pp. 997-999.

[6] *Sachkov, V.N.*, Combinatorial methods in discrete Mathematics. *Encyclopedia of mathematics and its applications 55.* Cambridge Press. 2008. P. 305.

[7] *Dashkova O. Yu.* On groups of finite normal rank. Algebra Discrete Math. 2002. 1, No. 1. P. 64-68.

[8] N. V. Maslova, D. O. Revin, On the Pronormality of Subgroups of Odd Index in Some Direct Products of Finite Groups, Journal of Algebra and Its Applications, doi: https://doi.org/10.1142/S0219498823500834

[9] *Ruslan V. Skuratovskii.* On commutator subgroups of Sylow 2-subgroups of the alternating group, and the commutator width in wreath products. / Ruslan V. Skuratovskii // European Journal of Mathematics. – 2021. – vol. 7, no. 1. – P. 353-373.