On three-dimensional equidistant pseudo-Riemannian spaces

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A pseudo-Riemannian space V_n with metric tensor g_{ij} is called equidistant if there exists in it a vector field $\phi_i \not\equiv 0$ (also called an equidistance vector) that satisfies the equation

$$\phi_{i,j} = \tau g_{ij}. \tag{1}$$

where τ is some invariant, and the comma "," is the sign of the covariant derivative in V_n. When $\tau \neq 0$ this is the equidistant space of the basic case, and when $\tau = 0$ it is the special case [1].

The integrability conditions of the basic equations (1) have the form

$$\phi_\alpha R^{\alpha}_{\cdot ijk} = \tau_{,k} g_{ij} - \tau_{,j} g_{ik}. \tag{2}$$

In equidistant spaces, the equidistant vector is proportional to the tensor $\tau_{,i}$. Since $\phi_i \not\equiv 0$, there exists a vector ξ_i such that the convolution $\phi_\alpha \xi^\alpha = 1$ and, then, from the integrability conditions (2) it is not difficult to obtain that

$$\tau_{,i} = B\phi_i, \;\; B \stackrel{def}{=} \tau_{,\alpha}\xi^\alpha. \tag{3}$$

From these same conditions, multiplying by g^{ij} and folding over the indices i, j, we obtain

$$\tau_{,i} = \frac{1}{(n-1)}\phi_\alpha R^{\alpha}_{\cdot i} \tag{4}$$

For $n = 3$ the curvature tensor in pseudo-Riemannian spaces has the following form [2]:

$$R_{ijkl} = R_{il}g_{jk} - R_{ik}g_{jl} + R_{jk}g_{il} - R_{jl}g_{ik} - \frac{R}{2}(g_{il}g_{jk} - g_{ik}g_{jl}). \tag{5}$$

Next, let's consider (4) in (3) for $n = 3$

$$\phi_\alpha R^{\alpha}_{\cdot i} = 2B\phi_i \tag{6}$$

Taking into account equidistance and (6), from (5) we obtain the following identity

$$\phi_k(g_{jl}(\frac{R}{2} - B) - R_{jl}) - \phi_l(g_{jk}(\frac{R}{2} - B) - R_{jk}) = 0. \tag{7}$$

From here we obtain the following form for the Ricci tensor

$$R_{jl} = \phi_j\phi_l\Phi + g_{jl}(\frac{R}{2} - B), \;\; \Phi \stackrel{def}{=} \xi_\alpha\xi_\beta R^{\alpha\beta} - \xi_\alpha\xi^\alpha(\frac{R}{2} - B) \tag{8}$$

and then we substitute this into (5) and obtain the form for the curvature tensor

$$R_{ijkl} = \Phi(\phi_i\phi_l g_{jk} - \phi_i\phi_k g_{jl} + \phi_j\phi_k g_{il} - \phi_j\phi_l g_{ik}) + (\frac{R}{2} - 2B)(g_{jk}g_{il} - g_{jl}g_{ik}). \tag{9}$$

Thus, a necessary condition is obtained for a pseudo-Riemannian space to be three-dimensional and equidistant. Formulas (8) and (9) allow us to more effectively investigate objects of these spaces, mappings, etc.

References

[1] V. A. Kiosak. On equidistant pseudo-riemannian spaces, *Mat. Stud.* 36 (2011), 21–25.
[2] Eisenhart L. P. *Riemannian geometry* - Princeton Univ. Press. - 1926. - 272p.