On the asymptotic behavior at infinity of solutions of the Beltrami equation with two characteristics

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Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let μ and $\nu\colon D\to\mathbb{C}$ be a measurable functions with $|\mu(z)|+|\nu(z)|<1$ a.e. (almost everywhere) in D. We study the *Beltrami equation with two characteristics*

$$f_{\overline{z}} \,=\, \mu(z)f_z+\nu(z)\overline{f_z}\quad \text{a.e.}\quad \text{in}\quad D, \tag{1}$$

where $f_{\overline{z}}=(f_x+if_y)/2$, $f_z=(f_x-if_y)/2$, $z=x+iy$, f_x and f_y are the partial derivatives of f by x and y, respectively. The functions μ and ν are called the *complex coefficients* and

$$K_{\mu,\,\nu}(z):=\frac{1+|\mu(z)|+|\nu(z)|}{1-|\mu(z)|-|\nu(z)|}$$

the *dilatation quotient* for the equation (1).

Picking $\nu(z)\equiv 0$ in (1), we arrive at the standard *Beltrami equation* of the form

$$f_{\overline{z}} \,=\, \mu(z)f_z. \tag{2}$$

For the equation (2) we set

$$K_{\mu}(z)\,=\,\frac{1+|\mu(z)|}{1-|\mu(z)|}.$$

Picking $\mu(z)\equiv 0$ in (1), we arrive at the *Beltrami equation of the second type*

$$f_{\overline{z}} \,=\, \nu(z)\overline{f_z}. \tag{3}$$

For the equation (3) we set

$$K_{\nu}(z)\,=\,\frac{1+|\nu(z)|}{1-|\nu(z)|}.$$

Let $z_0\in\mathbb{C}$ and $r>0$. We put $B(z_0,\,r)=\{z\in\mathbb{C}:|z-z_0|<r\}$.

We say that a function $\varphi\colon\mathbb{C}\to\mathbb{R}$ has a *global finite mean value* at the point $z_0\in\mathbb{C}$, abbr. $\varphi\in GFMV(z_0)$, if

$$\limsup_{R\to\infty}\frac{1}{\pi R^2}\int\limits_{B(z_0,R)}|\varphi(z)|\,dxdy<\infty.$$

For homeomorphism $f:\mathbb{C}\to\mathbb{C}$ we put

$$L_f(z_0,r)=\max_{|z-z_0|=r}|f(z)-f(z_0)|,\quad l_f(z_0,r)=\min_{|z-z_0|=r}|f(z)-f(z_0)|.$$

Theorem 1. *Let μ and $\nu\colon \mathbb{C}\to\mathbb{C}$ be a measurable functions with $|\mu(z)|+|\nu(z)|<1$ a.e. and $f\colon \mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation* (1)*, $z_0\in\mathbb{C}$. Assume that $K_{\mu,\nu}\in GFMV(\mathbb{C})$ and*

$$k_\infty = k_\infty(z_0) = \sup_{R\in(e,+\infty)} \frac{1}{\pi R^2}\int\limits_{B(z_0,R)} K_{\mu,\nu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

Picking $\nu(z)\equiv 0$ in Theorem 1, we arrive at the following statement.

Theorem 2. *Let $\mu\colon \mathbb{C}\to\mathbb{C}$ be a measurable function with $|\mu(z)|<1$ a.e. and $f\colon \mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation* (2)*, $z_0\in\mathbb{C}$. Assume that $K_\mu\in GFMV(\mathbb{C})$ and*

$$k_\infty = \sup_{R\in(e,+\infty)} \frac{1}{\pi R^2}\int\limits_{B(z_0,R)} K_{\mu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

Letting $\mu(z)\equiv 0$ in Theorem 1, we derive the following statement.

Theorem 3. *Let $\nu\colon \mathbb{C}\to\mathbb{C}$ be a measurable function with $|\nu(z)|<1$ a.e. and $f\colon \mathbb{C}\to\mathbb{C}$ be a homeomorphic $W^{1,1}_{\rm loc}$ solution of the Beltrami equation* (3)*, $z_0\in\mathbb{C}$. Assume that $K_\nu\in GFMV(\mathbb{C})$ and*

$$k_\infty = \sup_{R\in(e,+\infty)} \frac{1}{\pi R^2}\int\limits_{B(z_0,R)} K_{\nu}(z)\,dxdy,$$

then

$$\liminf_{R\to\infty}\frac{L_f(z_0,R)}{R^p}\geqslant c\,l_f(z_0,e),$$

where $p=\frac{2}{e^2k_\infty}$ and $c=e^{-\frac{4}{e^2k_\infty}}$.

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