SPACE-LIKE MINIMAL SURFACE IN MINKOWSKI SPACE AND THEIR GRASSMAN IMAGE

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There is a coordinate system in Minkowski space ${}^1R_4$ for which the metric of the space has the form $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$. Let the equation $\bar{r} = \bar{r}(u^1, u^2)$ defines a two-dimensional space-like surface F^2, the vectors $\bar{\xi}_1, \bar{\xi}_2$ are respectively its time-like and space-like normal vectors, and g_{ij}, L^k_{ij} are respectively the coefficients of the first and second quadratic forms. The number $H^k = \frac{1}{2} g^{ij} L^k_{ij}$ is called the mean curvature of the surface for the direction of the normal $\bar{\xi}_k$, and the vector $H = H^1\bar{\xi}_1 + H^2\bar{\xi}_2$ is the mean curvature vector. Space-like surfaces of Minkowski space with zero mean curvature vector will be called minimal surfaces, as in Euclidean space.

We plan to apply the concept of the indicatrix of the normal curvature of a surface to the study of its differential geometry, in particular, the question of the existence of such surfaces with some additional conditions on their Grassman image. The question of the existence of a time-like minimal surface with a constant curvature of its Grassman image was solved in work [3].

At point x on a space-like surface F^2 each direction $\bar{\tau} \in T_xF^2$ corresponds to a normal curvature vector $k(\bar{\tau}) = (Pr_{\bar{\xi}_1}, \bar{r}_{ss})\bar{\xi}_1 + (Pr_{\bar{\xi}_2}, \bar{r}_{ss})\bar{\xi}_2 = -(\bar{r}_{ss}, \bar{\xi}_1)\bar{\xi}_1 + (\bar{r}_{ss}, \bar{\xi}_2)\bar{\xi}_2 = -\frac{II^1}{ds^2}\bar{\xi}_1 + \frac{II^2}{ds^2}\bar{\xi}_2$, where $\bar{r}_{ss}$ is the curvature vector of the curve on the surface F^2 at the point x, which has the direction $\bar{\tau}$, and the scalar projections are defined by the formulas $Pr_{\bar{\xi}_i}, \bar{r}_{ss} = sign(\bar{\xi}_i^2)\frac{(\bar{r}_{ss}, \bar{\xi}_i)}{\sqrt{|\bar{\xi}_i^2|}}$. When direction $\bar{\tau}$ rotates in the tangent plane $\bar{\tau} \in T_xF^2$, the end $P(-\frac{II^1}{ds^2}; \frac{II^2}{ds^2})$ of the vector $\bar{r}_{ss}$ will form a curve, which we will call the indicatrix of normal curvature by analogy with Euclidean space [1].

Let us move on to such parameterization u^1, u^2 of the surface, for which the metric tensor has the form $g_{ij} = \delta_{ij}$. Next, we select a point (α, β), $\alpha = -\frac{L^1_{11} + L^1_{22}}{2}$, $\beta = -\frac{L^2_{11} + L^2_{22}}{2}$ in the plane N_x as the origin of the coordinate system. The geometric meaning of this transfer is to move to a raper with the origin in the center of the normal curvature indicatrix. Next, we choose normals $\bar{\xi}_1, \bar{\xi}_2$ parallel to the axes of the indicatrix, introduce the notations $\frac{L^1_{11} - L^1_{22}}{2} = a$, $L^2_{12} = b$, and obtain expressions for the coefficients of the second quadratic forms in the form $L^1_{11} = -(\alpha - a)$, $L^1_{12} = 0$, $L^1_{22} = -(\alpha + a)$, $L^2_{11} = \beta$, $L^2_{12} = b$, $L^2_{22} = \beta$.

The Grassmann image of two-dimensional surfaces is an important geometric characteristic of them. In [2] it is shown that the nondegenerate Grassmann image Γ^2 of a surface of Minkowski space is a two-dimensional surface $\bar{p} = \bar{p}(u^1, u^2)$, which belongs to the four-dimensional Grassmann submanifold $PG(2, 4)$ of the six-dimensional pseudo-Euclidean space ${}^3R_6$ of index 3. Tangent vectors to Γ^2 can be written in the form $\bar{p}_{u_i} = -L^1_{ik} g^{kl}[\bar{r}_l, \bar{\xi}_2] - L^2_{ik} g^{kl}[\bar{\xi}_1, \bar{r}_l], l = 1, 2$.

From the condition $g^{ij} L^k_{ij} = 0$ of minimality of the surface it follows that $\alpha = \beta = 0$. The metric of the Grassmann image of the minimal space-like surface of the space ${}^1R_4$ in the parameters of the normal curvature indicatrix has the form $ds^2 = (a^2 - b^2)^2 g_{11} g_{22} du^1 du^2$, and therefore the Grassmann image of the surface is also a space-like surface. The formula for the sectional curvature of the Grassmann image has the form $\overline{K} = -1 + \frac{4a^2b^2}{(a^2 - b^2)^2}$ and therefore it can take on values from the interval $(-1; +\infty)$.

To solve the problem of the existence of space-like minimal surfaces with a non-degenerate Grassmann image of constant curvature $\overline{K}$, it is necessary to prove that under this condition the system of Gauss-Kodazzi-Ricci equations

$$
\begin{cases}
R_{1212} = (a^2 - b^2)^2 g_{11} g_{22}, \\
(a g_{11})'_{u^2} = b\sqrt{g_{11} g_{22}}\mu_{12/1}, \\
(a g_{22})'_{u^1} = -b\sqrt{g_{11} g_{22}}\mu_{12/2}, \\
(b\sqrt{g_{11} g_{22}})'_{u^1} = -a g_{11}\mu_{12/2}, \\
(b\sqrt{g_{11} g_{22}})'_{u^2} = a g_{11}\mu_{12/1}, \\
(\mu_{12/1})'_{u^2} - (\mu_{12/2})'_{u^1} = -2ab\sqrt{g_{11} g_{22}},
\end{cases} \tag{1}
$$

is compatible. Here $\mu_{12/i}$ are the torsion coefficients.

References

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