On convergence of homeomorphisms with inverse Poletsky inequality

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Below $dm(x)$ denotes the element of the Lebesgue measure in $\mathbb{R}^n$. Everywhere further the boundary ∂A of the set A and the closure $\overline{A}$ should be understood in the sense of the extended Euclidean space $\overline{\mathbb{R}^n}$. Recall that, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for the family Γ of paths γ in $\mathbb{R}^n$, if the relation

$$\int\limits_{\gamma} \rho(x)\,|dx| \geqslant 1 \tag{1}$$

holds for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm}\Gamma$. The *modulus* of Γ is defined by the equality

$$M(\Gamma) = \inf\limits_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^n(x)\,dm(x)\,. \tag{2}$$

Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{ y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2 \}\,. \tag{3}$$

Given $x_0 \in \mathbb{R}^n$, we put $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{B}^n = B(0, 1)$, $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$. A mapping $f : D \to \mathbb{R}^n$ is called *discrete* if the pre-image $\{f^{-1}(y)\}$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and *open* if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. Given sets $E, F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$ we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for $t \in (a, b)$. Given a mapping $f : D \to \mathbb{R}^n$, a point $y_0 \in \overline{f(D)} \setminus \{\infty\}$, and $0 < r_1 < r_2 < r_0 = \sup\limits_{y \in f(D)} |y - y_0|$, we denote by $\Gamma_f(y_0, r_1, r_2)$ a family of all paths γ in D such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$. Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function. We say that f *satisfies the inverse Poletsky inequality at a point* $y_0 \in \overline{f(D)} \setminus \{\infty\}$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leqslant \int\limits_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|)\,dm(y) \tag{4}$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\,dr \geqslant 1\,. \tag{5}$$

The relations (4) are proved for different classes of mappings, see e.g. [1].

Set $q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(y_0, r)} Q(y)\,d\mathcal{H}^{n-1}(y)$, where ω_{n-1} denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. We say that a function $\varphi : D \to \mathbb{R}$ has a *finite mean oscillation* at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if $\limsup\limits_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0, \varepsilon)} |\varphi(x) - \overline{\varphi}_{\varepsilon}|\,dm(x) < \infty$, where $\overline{\varphi}_{\varepsilon} = \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0, \varepsilon)} \varphi(x)\,dm(x)$ and Ω_n is the volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A\subset \overline{D}$, write $\varphi\in FMO(A)$, if φ has a finite mean oscillation at any point $x_0\in A$. Let h be a chordal metric in $\overline{{\mathbb R}^n}$,

$$h(x,\infty)=\frac{1}{\sqrt{1+{|x|}^2}}\,,\quad h(x,y)=\frac{|x-y|}{\sqrt{1+{|x|}^2} \sqrt{1+{|y|}^2}}\,, \qquad x\ne \infty\ne y\,,$$

and let $h(E):=\sup\limits_{x,y\in E}h(x,y)$ be a chordal diameter of a set $E\subset \overline{{\mathbb R}^n}$ (see, e.g., [2, Definition 12.1]).

Theorem 1. *Let D be a domain in ${\mathbb R}^n$, $n\geqslant 2$, and let $f_j:D\rightarrow {\mathbb R}^n$, $j=1,2,\ldots,$ be a sequence of homeomorphisms that converges to some mapping $f:D\rightarrow \overline{{\mathbb R}^n}$ locally uniformly in D by the metric h, and satisfy the relations (4)–(5) in each point $y_0\in \overline{{\mathbb R}^n}$. Assume that one of two conditions holds:*

1) $Q\in FMO(\overline{{\mathbb R}^n})$, or

2) for any $y_0\in \overline{{\mathbb R}^n}$ there exist $\varepsilon_1(y_0)>0$ and $\delta(y_0)>0$ such that

$$\int\limits_{\varepsilon}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}<\infty \qquad \forall\ \varepsilon\in(0, \varepsilon_1(y_0)), \qquad \int\limits_{0}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}=\infty\,. \tag{6}$$

Then f is either a homeomorphism $f:D\rightarrow {\mathbb R}^n$, or a constant $c\in \overline{{\mathbb R}^n}$.

Here the conditions mentioned above for $y_0=\infty$ must be understood as conditions for the function $\widetilde{Q}(y):=Q(y/|y|^2)$ at the origin. We should note that the second condition in (6) is not only a sufficient but also a necessary condition in Theorem 1. The following conclusion holds.

Theorem 2. *Let $Q:{\mathbb R}^n\rightarrow [0, \infty]$ be locally integrable function such that*

$$\int\limits_{0}^{\delta(y_0)}\frac{dt}{tq_{y_0}^{\frac{1}{n-1}}(t)}<\infty$$

for some $y_0\in {\mathbb R}^n$ and $\delta(y_0)>0$. Then there exists a sequence of homeomorphisms $f_j:D\rightarrow {\mathbb R}^n$, $j=1,2,\ldots,$ satisfying the relations (4)–(5) at y_0 which converges to some mapping $f:D\rightarrow \overline{{\mathbb R}^n}$ locally uniformly in D by the metric h, which is neither a homeomorphism nor a constant.

The results mentioned above are published in [3].

References

[1] Martio O., Ryazanov V., Srebro U. and Yakubov E. *Moduli in Modern Mapping Theory.* Springer Science + Business Media, LLC : New York, 2009.

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