Are Totally Convex Surfaces Area Minimizing? (A note on definitions and counterexamples)

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We investigate the claim that a compact surface Σ with boundary $\partial\Sigma$, embedded in a manifold M, is area minimizing among all surfaces with the same boundary if it satisfies a "total convexity" property. This problem connects methods from geometric analysis and algebraic geometry. We clarify two main interpretations of total convexity: intrinsic (geodesic) and extrinsic (ambient). Under the intrinsic definition, we demonstrate the claim is false using the counterexample of a spherical cap. For the extrinsic definition which implies Σ is a convex domain in a totally geodesic submanifold $P \subset M$ the claim holds in Euclidean space (proven via calibration using differential forms) and is generally true for hypersurfaces in Riemannian manifolds where stability analysis is more tractable. However, it can fail in higher codimension due to more complex stability criteria. We provide justifications, citing known counterexamples from minimal surface theory, particularly those in normed spaces, which underscore the subtleties. Connections to rigidity problems are briefly explored, highlighting how specific geometric and algebraic structures might enforce uniqueness.