Anabelian geometry in arithmetic topology

Jun Ueki
(Department of Mathematics, Ochanomizu University; Tokyo, Japan)
E-mail: uekijun46@gmail.com

This talk is based on a joint work with Nadav Gropper and Yi Wang [GUW25].

The analogy between knots and primes, or 3-manifolds and the ring of integers of number fields, has been systematically developed by Mazur [Maz64, Maz12], Kapranov [Kap95], Reznikov [Rez97, Rez00], Morishita [Mor02, Mor12, Mor24], Kim [Kim20], and others. In their spirit of *arithmetic topology*, we have formulated in [Nii14, NU19] an analogue of Artin–Takagi–Chevalley's *idelic class field theory* that sums up all local theories to describe all abelian branched covers of a 3-manifold M endowed with a certain infinite link $\mathcal{K}$. Successive studies are [Mih19, NU23, Tas25b, Tas25a]. In addition, analogues of the set of all primes have been studied in [Maz12, McM13, Uek20, Uek21a, Uek21b].

Extending this context, we may discuss an analogue of so-called *anabelian geometry*, whose initial fundamental result is the classical *Neukirch–Uchida theorem* stated as follows.

Theorem 1 (Neukirch [Neu69b, Neu69a], Uchida [Uch76], see also [NSW08, Theorem 12.2.1]). *Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Let E, F be number fields, that is, finite extensions of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$. If there is an isomrphism $\varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/E) \overset{\cong}{\to} \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of topological groups, then there uniquely exists a natural isomorphism $E \overset{\cong}{\to} F$, that is, there is a unique $\sigma \in \mathrm{Aut}\,\overline{\mathbb{Q}}$ such that $F = \sigma(E)$ and σ induces φ.*

In the proof, the Hilbert ramification theory for infinite Galois extensions, the Poiteau–Tate duality, and the Chebotarev density theorem play key roles. One of the main steps is to prove the following.

Theorem 2 ([NSW08, Theorem 12.2.5]). *Let $F/\mathbb{Q}$ be a finite Galois extension and $E/\mathbb{Q}$ a finite extension. If all primes $p \in \mathbb{Q}$ with a prime factor of degree 1 in $E/\mathbb{Q}$ completely decompose in $F/\mathbb{Q}$, then $F \subset E$.*

Now let M be an oriented connected closed 3-manifold with a base point b_M and let $\mathcal{K} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} K_i$ be an infinite link consisting of countably many tame components. Let $\mathrm{Cov}(M, \mathcal{K})$ denote the set of all branched covers branched along finite sublinks of $\mathcal{K}$. We define *the absolute Galois group* of $(M, \mathcal{K})$ by $\mathrm{Gal}(M, \mathcal{K}) = \varprojlim_{h \in \mathrm{Cov}(M,\mathcal{K})} \mathrm{Gal}\,h = \varprojlim_{L \subset \mathcal{K}} \widehat{\pi}_1(M - L)$, where $\widehat{\pi}_1$ denotes the profinite completion of π_1. Then, we may formulate *the Hilbert ramification theory* for pro-covers [GUW25]. Suppose in addition that $\mathcal{K}$ obeys *the Chebotarev law.* Then it turns out that for any $h \in \mathrm{Cov}(M, \mathcal{K})$, the inverse image $h^{-1}(\mathcal{K})$ is again Chebotarev [GUW25]. An analogue of Theorem 1 may be stated as follows.

Theorem 3 ([GUW25]). *Let the setting be as above. Let G_1, G_2 be open subgroups of $\mathrm{Gal}(M, \mathcal{K})$ and let $h_1, h_2 \in \mathrm{Cov}(M, \mathcal{K})$ denote the corresponding branched covers. If there is an isomorphism $\varphi : G_1 \overset{\cong}{\to} G_2$ of topological groups, then there uniquely exists a natural isomorphism $h_1 \cong h_2$ of branched covers, that is, there is a unique $\sigma \in \mathrm{Gal}(M, \mathcal{K})$ such that $h_2 \circ \sigma = h_1$ and σ induces φ.*

An analogue of the key step is as follows.

Theorem 4 ([GUW25]). *Let $h_1, h_2 \in \mathrm{Cov}(M, \mathcal{K})$ and suppose that h_1 is Galois. If all knots $K \subset \mathcal{K}$ whose inverse image $h_2^{-1}(K)$ has a component of covering degree 1 in h_2 completely decompose in h_1, then h_1 is a subcover of h_2.*

Once the theorem's statement comes into view, in the context of research aiming to systematize analogies, numerous problems to be addressed in the future become apparent. In the topology side, the classical Mostow rigidity assures that hyperbolic manifolds are determined by their fundamental groups. In addition, in recent days, profinite rigidity has been of great interest [Rei18, BJZR23]. But we believe that rigidity for such a large group $\mathrm{Gal}(M, \mathcal{K})$ is a new viewpoint and would be of interest, even away from the context of the analogy.

References

[BJZR23] Martin Bridson, Andrei Jaikin-Zapirain, and Alan W. Reid, *A list of problems suggested by participants of Workshop on Profinite Rigidity*, `https://www.icmat.es/RT/2023/GGTLDGT/week_3_problem_list.pdf`, 2023.

[GUW25] Nadav Gropper, Jun Ueki, and Yi Wang, *Neukirch–Uchida theorem for knots (tentative)*, in preparation, 2025.

[Kap95] M. M. Kapranov, *Analogies between the Langlands correspondence and topological quantum field theory*, Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), Progr. Math., vol. 131, Birkhäuser Boston, Boston, MA, 1995, pp. 119–151. MR 1373001 (97c:11069)

[Kim20] Minhyong Kim, *Arithmetic Chern-Simons theory I*, Galois covers, Grothendieck-Teichmüller Theory and Dessins d'Enfants, Interactions between geometry, topology, number theory and algebra, Leicester, UK, June 2018 (Frank Neumann and Sibylle Schroll, eds.), Springer Proceedings in Mathematics & Statistics, vol. 330, Springer, Cham, 2020, (cf. arXiv:1510.05818). MR 4166922

[Maz64] Barry Mazur, *Remarks on the Alexander polynomial*, `https://bpb-us-e1.wpmucdn.com/sites.harvard.edu/dist/a/189/files/2023/01/Remarks-on-the-Alexander-Polynomial.pdf`, 1963–64.

[Maz12] ———, *Primes, Knots and Po*, Lecture notes for the conference "Geometry, Topology and Group Theory" in honor of the 80th birthday of Valentin Poenaru, `https://bpb-us-e1.wpmucdn.com/sites.harvard.edu/dist/a/189/files/2023/01/Primes-Knots-and-Po.pdf`, July 2012.

[McM13] Curtis T. McMullen, *Knots which behave like the prime numbers*, Compos. Math. **149** (2013), no. 8, 1235–1244. MR 3103063

[Mih19] Tomoki Mihara, *Cohomological approach to class field theory in arithmetic topology*, Canad. J. Math. **71** (2019), no. 4, 891–935. MR 3984024

[Mor02] Masanori Morishita, *On certain analogies between knots and primes*, J. Reine Angew. Math. **550** (2002), 141–167. MR 1925911 (2003k:57008)

[Mor12] ———, *Knots and primes*, Universitext, Springer, London, 2012, An introduction to arithmetic topology. MR 2905431

[Mor24] ———, *Knots and primes*, Universitext, Springer Singapore, 2024, An introduction to arithmetic topology, 2nd edition.

[Neu69a] Jürgen Neukirch, *Kennzeichnung der endlich-algebraischen Zahlkörper durch die Galoisgruppe der maximal auflösbaren Erweiterungen*, J. Reine Angew. Math. **238** (1969), 135–147. MR 258804

[Neu69b] ———, *Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper*, Invent. Math. **6** (1969), 296–314. MR 244211

[Nii14] Hirofumi Niibo, *Idèlic class field theory for 3-manifolds*, Kyushu J. Math. **68** (2014), no. 2, 421–436. MR 3243372

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026 (2008m:11223)

[NU19] Hirofumi Niibo and Jun Ueki, *Idèlic class field theory for 3-manifolds and very admissible links*, Trans. Amer. Math. Soc. **371** (2019), no. 12, 8467–8488. MR 3955553

[NU23] ———, *A Hilbert reciprocity law on 3-manifolds*, Res. Math. Sci. **10** (2023), no. 1, Paper No. 3, 8. MR 4519217

[Rei18] Alan W. Reid, *Profinite rigidity*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 1193–1216. MR 3966805

[Rez97] Alexander Reznikov, *Three-manifolds class field theory (homology of coverings for a nonvirtually b_1-positive manifold)*, Selecta Math. (N.S.) **3** (1997), no. 3, 361–399. MR 1481134 (99b:57041)

[Rez00] ———, *Embedded incompressible surfaces and homology of ramified coverings of three-manifolds*, Selecta Math. (N.S.) **6** (2000), no. 1, 1–39. MR 1771215 (2001g:57035)

[Tas25a] Hirotaka Tashiro, *On genus theory for 3-manifolds in arithmetic topology*, preprint. arXiv:, 2025.

[Tas25b] ———, *On Hasse norm principle for 3-manifolds in arithmetic topology*, Res. Math. Sci. **12** (2025), no. 2, Paper No. 27. MR 4884597

[Uch76] Kôji Uchida, *Isomorphisms of Galois groups*, J. Math. Soc. Japan **28** (1976), no. 4, 617–620. MR 432593

[Uek20] Jun Ueki, *Olympic links in a Chebotarev link*, Proc. Int. Geom. Cent. **13** (2020), no. 4, 40–49. MR 4237777

[Uek21a] ———, *Chebotarev links are stably generic*, Bull. Lond. Math. Soc. **53** (2021), no. 1, 82–91. MR 4224512

[Uek21b] ———, *Modular knots obey the chebotarev law*, preprint. arXiv:2105.10745, May 2021.