On geometries of Kac-Moody groups and Symbolic Computations

Vasyl Ustimenko
(Institute of telecommunications and global information space, NAS of Ukraine, Kyiv)
E-mail: `vasylustimenko@yahoo.pl`

Let K be a commutative ring with the unity. Jordan-Gauss graph over K is an incidence structures with partition sets P (points) and L (lines) isomorphic to affine spaces V_1 and V_2 over K such that the incidence relation is given by special quadratic equations over the commutative ring K with unity such that the neighbour of each vertex is defined by the system of linear equation given in its row-echelon form.

We assume that V_i, $i = 1, 2$ are finite dimensional spaces of kind K^n or infinite dimensional affine spaces formed tu tuples with finite support.

Let Γ be an incidence system with partition sets Γ_i, $i = 1, 2, \ldots, m$ and incidence relation I. We say that equivalence τ on Γ is Jordan-Gauss equivalence over K if each equivalence class is an affine space over this ring, each Γ_i is a union of these equivalence classes and the restriction of I on the union of two such classes consisting of elements of different types is a Jordan-Gauss graph or empty relation.

Theorem 1. *Let F be a field, $G(F)$ be a Kac-Moody group and $\Gamma(G(F))$ be a Kac - Moody geometry of $G(F)$. Then there is Jordan-Gauss equivalence on $\Gamma(G(F))$ defined over F. Totality of equivalence classes of this relation is in one to one correspondence with elements of corresponding Weyl geometry.*

The theorem is the corollary of the results presented in [1], [2].

Let B^+ and B^- be Borel subgroups containing root subgroups corresponding positive and negative roots respectively. Let P_i, $i = 1, 2, \ldots, m$ are standard maximal parabolic subgroups, i. e maximal subgroups of G containing B^+. The geometry $\Gamma(G(F))$ is the disjoint union of $(G(F) : P_i)$ with the type function $t(gP_i) = i$ and incidence relation $I : \alpha I \beta$ if and only if $\alpha \cap \beta$ is not an empty set. Orbits of B^- form the classes of Jordan-Gauss equivalence relation.

Jordan-Gauss graph is the special case of linguistic graph given by the following way. We identify points with tuples of kind $(x) = (x_1, x_2, \ldots, x_n, \ldots)$ and lines with tuples $[y] = [y_1, y_2, \ldots, y_n, \ldots]$. Brackets and parenthesis are convenient to distinguished type of the vertex of the graph. Elements (x) and $[y]$ are incident $(x)I[y]$ if and only if the following relations hold.

$a_1 x_{s+1} - b_1 y_{r+1} = f_1(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_r)$,
$a_2 x_{s+2} - b_2 y_{r+2} = f_2(x_1, x_2, \ldots, x_s, x_{s+1}, y_1, y_2, \ldots, y_r, y_{r+1})$,
$\ldots$
$a_m x_{s+m} - b_m y_{r+m} = f_m(x_1, x_2, \ldots, x_s, x_{s+1}, \ldots, x_{s+m-1}, y_1, y_2, \ldots, y_r, y_{r+1}, \ldots, y_{r+m-1})$
...

where a_j and b_j, $j = 1, 2, \ldots, m$ are not zero divisors, and f_j are multivariate polynomials with coefficients from K.

Linguistic graph given by the written above equations is Jordan – Gauss graph if the map sending the pair $((x_1, x_2, \ldots, x_n, \ldots), (y_1, y_2, \ldots, y_n, \ldots))$ to $(f_1, f_2, \ldots, f_m, \ldots)$ is a bilinear one.

We use the interpretations of geometries $\Gamma(G(F))$ to define their analogues $\Gamma(G(K))$ where K is arbitrary commutative ring with unity. The walks on incidence structures $\Gamma(G(K)$ and natural colourings of their Jordan - Gauss graphs are used for the design of explicit constructions of groups supporting the following statement of Computational Algebraic Geometry.

Theorem 2. *Let K be commutative ring with unity. For each positive integer n, d, $d \geq 2$ and rational parameter $s \geq 0$ there is a subgroup H of affine Cremona semigroup of all endomorphisms of $K[x_1, x_2, \ldots, x_n]$ such that maximal degree of representative of H is d and the densities of elements from H are of size $O(n^s)$.*

Recall that degree and density of endomorphism F of $K[x_1, x_2, \ldots, x_n]$ is defined as maximal values of degrees and densities of standard forms of polynomials $F(x_i)$, $i = 1, 2, \ldots, n$. For each commutative ring K with unity and selected positive integer d, $d > 1$ and rational parameter s, $d \geq s > 0$ we construct polynomial bijective map F of K^n onto K^n of degree d, density $O(n^s)$ with the computational accelerator T which is the piece of information such that its knowledge allows to compute the reimage of F in time $O(n^2)$.

During the talk some applications of these results to Computational Algebraic Geometry the Theory of Communications will be described (see [3]).

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References

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