Hidden symmetry-like structures of dispersionless Nizhnik equation

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In [3], L. P. Nizhnik studied the integration of multidimensional nonlinear equations using the inverse scattering method. Therein, the system

$$w_t = k_1 w_{xxx} + k_2 w_{yyy} + 3(v^1 w)_x + 3(v^2 w)_y, \quad v^1_y = k_1 w_x, \quad v^2_x = k_2 w_y \tag{1}$$

known now as the Nizhnik system was introduced. It has two parameters, k_1 and k_2 with $(k_1, k_2) \neq (0,0)$, but the only thing that matters is whether the product $k_1 k_2$ is zero (the asymmetric case) or not (the symmetric case). Introducing potentials in the system (1) and/or taking limits with respect to a small scaling parameter, we can derive various related models,

- the (symmetric potential) Nizhnik equation $u_{txy} = u_{xxxxy} + u_{xyyyy} + 3(u_{xx}u_{xy})_x + 3(u_{yy}u_{xy})_y$,
- the asymmetric (potential) Nizhnik equation $u_{ty} = u_{xxxy} + 3(u_x u_y)_x$ (also called the Boiti–Leon–Manna–Pempinelli equation [1]),
- the (symmetric) dispersionless Nizhnik system $w_t = (v^1 w)_x + (v^2 w)_y$, $v^1_y = w_x$, $v^2_x = w_y$,
- the (symmetric potential) dispersionless Nizhnik equation
$$u_{txy} = (u_{xx}u_{xy})_x + (u_{xy}u_{yy})_y, \tag{2}$$
- the asymmetric dispersionless Nizhnik system $w_t = (v^1 w)_x$, $v^1_y = w_x$,
- the asymmetric (potential) dispersionless Nizhnik equation $u_{ty} = (u_x u_y)_x$.

The results from [2] on point symmetries of the equation (2) created a basis for a comprehensive classification of its Lie reductions to partial differential equations with two independent variables and to ordinary differential equations, which was carried out in [5]. The list of inequivalent one-dimensional subalgebras of the maximal Lie invariance pseudoalgebra $\mathfrak{g}$ of (2) presented in [5] includes, in particular, the family of subalgebras $\mathfrak{s}^{\rho}_{1.3} = \langle P^x(1) + P^y(\rho) \rangle$, where $\rho = \rho(t)$ is an arbitrary smooth function of t satisfying the inequalities $\rho(t) \neq 0$ for all t in its domain and $\rho \not\equiv 1$ on any open interval within that domain. The optimal ansatzes constructed with respect to these subalgebras reduce the equation (2) to partial differential equations in two independent variables that share the same form

$$w_{122} + w_{22} w_{222} = 0. \tag{3}$$

It is easy to see that the substitution $w_{22} = h$ maps the equation (3) to the inviscid Burgers equation

$$h_1 + h h_2 = 0. \tag{4}$$

The equation (3) is the most interesting and fruitful submodel of (2), in particular, in the sense of its relation to hidden symmetry-like objects of (2). In [4], we found all local symmetry-like objects associated with the equation (3), including generalized symmetries, cosymmetries, conservation-law characteristics and conservation laws, and most of them are hidden for (2). This represents the first comprehensive study of such objects for a submodel of a well-known system of differential equations. Complete descriptions even of particular kinds of such objects in nontrivial cases exist in the literature only for a minor part of these systems themselves, not to mention submodels. Moreover, a complete description of all the local symmetry-like objects of a model in a single paper is rather exceptional.

Standard techniques like recursion operators and the estimation of the dimension of the space of objects in question up to an arbitrary fixed order do not work for the equation (3). Even the best computer packages for finding local symmetry-like objects such as Jets and GeM for Maple are inefficient at computing such objects for this equation even at low orders, starting from order three. This can be explained by the fact that for local symmetry-like objects of any specific kind, the corresponding space of them for the equation (3) is of complicated structure. In particular, they are parameterized by functions of arbitrary finite number of arguments that are cumbersome differential expressions.

To illustrate the above claims, here we present only the description of generalized symmetries of (3).

Theorem 1. *A differential function $f\{w\}$ is the characteristic of a generalized symmetry of the equation* (3) *if and only if it is a linear combination of the differential functions*

$$w_{1,0},\quad z_1w_{1,0}+w_{0,0},\quad z_1^2w_{1,0}+z_1z_2w_{0,1}-z_1w_{0,0}-\frac16z_2^3,$$

$$w_{0,1},\quad 2z_1w_{0,1}-z_2^2,\quad z_2w_{0,1}-3w_{0,0},\quad \breve g,\quad z_2g,\quad \frac{w_{0,2}}{w_{1,2}\theta^2}(\breve f_{w_{0,2}}-\theta^{k+1}\breve f_{\theta^k})+\breve f,$$

$$\frac{(w_{0,2})^3}{2w_{1,2}}\frac{(\theta^1)^2}{\theta^2}+\frac16z_1^2(w_{0,2})^3+z_1w_{1,0},\quad \frac23\frac{(w_{0,2})^4}{w_{1,2}}\frac{(\theta^1)^2}{\theta^2}+\frac16z_1^2(w_{0,2})^4+z_2w_{1,0}-z_2^2\zeta^{10},$$

$$2\frac{(w_{0,2})^3}{w_{1,2}}\frac{(\theta^1)^2}{\theta^2}\theta^0+\frac23z_1^2(w_{0,2})^3\theta^0+\frac16z_1^3(w_{0,2})^4+z_2w_{0,0}-z_2^2w_{0,1}-z_1z_2w_{1,0}+z_1z_2^2\zeta^{10}.$$

Here $w_{k,l}:=\partial^{k+l}w/\partial z_1^k\partial z_2^l$, g and $\breve g$ are arbitrary functions of z_1 and a finite number of ζ^{ik}, the $\breve f$ is an arbitrary function of $w_{0,2}$ and a finite number of θ^k, $k,l\in\mathbb N_0$, $i=1,2$,

$$\zeta^{ik}:=\mathrm D_1^kI^i,\quad \theta^k:=\left(\frac{w_{0,2}}{w_{1,2}}\hat{\mathrm D}_2\right)^k(z_2-w_{0,2}z_1),$$

$$I^1:=w_{1,1}+\frac12(w_{0,2})^2,\quad I^2:=w_{2,0}-\frac13(w_{0,2})^3-z_2(w_{2,1}+w_{0,2}w_{1,2})=w_{2,0}-\frac13(w_{0,2})^3-z_2\mathrm D_1I^1,$$

$\mathrm D_1$ *and* $\mathrm D_2$ *denote the operators of total derivatives with respect to the variables z_1 and z_2, and $\hat{\mathrm D}_2$ is the restriction of* $\mathrm D_2$ *to solutions of* (3),

$$\hat{\mathrm D}_2=\partial_{z_2}+w_{0,1}\partial_{w_{0,0}}+\left(\zeta^{10}-\frac12(w_{0,2})^2\right)\partial_{w_{1,0}}+w_{0,2}\partial_{w_{0,1}}-\hat{\mathrm D}_1^k\left(\frac{w_{1,2}}{w_{0,2}}\right)\partial_{w_{k,2}}.$$

References

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