Motivic Hilbert zeta functions of curve singularities and related invariants

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Let C and p be a reduced singular curve over $\mathbb{C}$ and its singular point respectively. We refer to the germ of C at p as a curve singularity and denote it by (C,p). Let $K_0(\mathrm{Var}_{\mathbb{C}})$ be the Grothendieck ring of complex algebraic varieties.

For a reduced curve singularity (C,p), the *motivic Hilbert zeta function* with support at p is defined as

$$Z^{\mathrm{Hilb}}_{C,p}(t) := \sum_{l=0}^{\infty}[C_p^{[l]}]t^l \in 1+tK_0(\mathrm{Var}_{\mathbb{C}})[[t]] \tag{1}$$

where $C_p^{[l]}$ consits of length l subschemes of C supported at p. It is known that $Z^{\mathrm{Hilb}}_{C,p}(t)$ is rational (see [1]). We refer to $C_p^{[l]}$ as *the punctual Hilbert scheme of degree* l for the given curve singularity (C,p). In this talk, we consider the following assumtion:

Assumption 1. For a given curve singularity (C,p), any punctual Hilbert scheme $C_p^{[l]}$ admits an affine cell decomposition.

Remark 2. It is known that any irreducible plane curve singularity with one Puiseux pair satisfies Assumption 1 (see [6]).

Let $\mathbb{L}$ denote the class of the affine line $\mathbb{A}^1$ in $K_0(\mathrm{Var}_{\mathbb{C}})$.

Lemma 3. *Let (C,p) be a reduced curve singularity that satisfies Assumption 1. Then the class $[C_p^{[l]}]$ in $K_0(\mathrm{Var}_{\mathbb{C}})$ is a polynomial in $\mathbb{C}[\mathbb{L}]$. Furthermore, the Euler number $\chi(C_p^{[l]})$ is equal to the number of affine cells of $C^{[l]}$.*

By Lemma 3, we see that the motivic Hilbert zeta function (1) is an element of $\mathbb{C}[\mathbb{L}][[t]]$ under Assumption 1. Therefore, instead of $Z^{\mathrm{Hilb}}_{C,p}(t)$, we use the notation $Z^{\mathrm{Hilb}}_{C,p}(t,\mathbb{L})$.

Theorem 4. *Let (C,p) be a reduced curve singularity. If Assumption 1 holds, then we have*

$$Z^{\mathrm{Hilb}}_{C,p}(q,1) = \sum_{l=0}^{\infty}\chi(C_p^{[l]})q^l$$

where $\chi(C_p^{[l]})$ is the Euler number of $C_p^{[l]}$.

Let Γ be a semigroup and let $\mathrm{Mod}(\Gamma)$ denote the set of all Γ-semimodules. For a Γ-semimodule Δ, we define its codimension by $\mathrm{codim}(\Delta) := \#(\Gamma\setminus\Delta)$. The *generating function $I(\Gamma;q)$ of Γ-semimodules* is defined to be

$$I(\Gamma;q) := \sum_{\Delta\in\mathrm{Mod}(\Gamma)} q^{\mathrm{codim}(\Delta)}.$$

Theorem 5. *For an irreducible curve singularity (C,p) with one Puiseux pair, the following realtion holds*:

$$Z^{\mathrm{Hilb}}_{C,p}(q,1) = I(\Gamma;q)$$

Using our results, we clarify the relations among Motivic Hilbert zeta functions and other invariants. Below we focus on reduced plane curve singularities. Let $P(L_{C,p})$ be the HOMFLY polynomial of the oriented link $L_{C,p}$ associated with (C,p). The following relation was conjectured by Oblomkov and Shende in [4] and was finally proved by Maulik in [3]:

$$\sum_{l=0}^{\infty} \chi(C_p^{[l]}) q^{2l} = \left(\frac{q}{a}\right)^{\mu-1} P(L_{C,p})\Big|_{a=0} \tag{2}$$

On the other hand, Shende [5] also proved the relation

$$\sum_{l=0}^{\infty} \chi(C_p^{[l]}) q^{l} = \sum_{l=0}^{\delta} q^{\delta-l}(1-q)^{2h-1}\mathrm{deg}_p \mathbb{V}_h \tag{3}$$

where δ is the delta invariant of (C,p) and $\mathbb{V}_h$'s are the severi strata of the miniversal deformation of (C,p).

Consequently, the following fact follows from Theorem 4 and 5, along with the relations (2) and (3).

Theorem 6. *Here notations remain the same as above. If (C,p) is an irreducible plane curve singularity with one Puiseux pair, then we have*

$$Z_{C,p}^{\mathrm{Hilb}}(q^2,1) = I(\Gamma;q^2) = \left(\frac{q}{a}\right)^{\mu-1} P(L_{C,p})\Big|_{a=0}, \tag{4}$$

$$Z_{C,p}^{\mathrm{Hilb}}(q,1) = I(\Gamma;q) = \sum_{l=0}^{\delta} q^{\delta-l}(1-q)^{2h-1}\mathrm{deg}_p \mathbb{V}_h. \tag{5}$$

Remark 7. The equivalence of the HOMFLY polynomial and the generating function of Γ-semimodules $I(\Gamma;q)$ in (4) was pointed out by Chavan ([2]).

References

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