Balayage in minimum Riesz energy problems with external fields

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This talk deals with a minimum energy problem in the presence of external fields on $\mathbb{R}^n$, $n \geqslant 2$, the energy being evaluated with respect to the α-Riesz kernel $\kappa_\alpha(x, y) := |x - y|^{\alpha - n}$, where $\alpha \in (0, n)$ and $\alpha \leqslant 2$. (Here $|x - y|$ is the Euclidean distance between $x, y \in \mathbb{R}^n$.) For precise formulations, we denote by $\mathfrak{M}$ the linear space of all (real-valued Radon) measures μ on $\mathbb{R}^n$, equipped with the *vague* topology of pointwise convergence on the continuous functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ of compact support, and by $\mathfrak{M}^+$ the cone of all positive $\mu \in \mathfrak{M}$. Given $\mu, \nu \in \mathfrak{M}$, we define the *mutual energy* and *potential* by means of

$$I(\mu, \nu) := \int \kappa_\alpha(x, y)\, d(\mu \otimes \nu)(x, y) \quad \text{and} \quad U^\mu(x) := \int \kappa_\alpha(x, y)\, d\mu(y), \quad x \in \mathbb{R}^n,$$

respectively, provided the value on the right is well defined as a finite number of $\pm\infty$. For $\mu = \nu$, $I(\mu, \nu)$ defines the *energy* $I(\mu) := I(\mu, \mu)$. A crucial fact is that κ_α is *strictly positive definite* in the sense that for any $\mu \in \mathfrak{M}$, $I(\mu)$ is $\geqslant 0$ whenever defined, and moreover $I(\mu) = 0 \iff \mu = 0$. This implies that all $\mu \in \mathfrak{M}$ with $I(\mu) < \infty$ form a pre-Hilbert space $\mathcal{E}$ with the inner product $\langle \mu, \nu \rangle := I(\mu, \nu)$ and the norm $\|\mu\| := \sqrt{I(\mu)}$. The topology on $\mathcal{E}$ defined by $\|\cdot\|$ is said to be *strong*. Moreover, κ_α is *perfect*, which means that the cone $\mathcal{E}^+ := \mathcal{E} \cap \mathfrak{M}^+$ is *strongly complete*, while the strong topology on $\mathcal{E}^+$ is *finer* than the induced vague topology on $\mathcal{E}^+$. (See Landkof's book [3] and historical notes therein.)

Fixing $A \subsetneqq \mathbb{R}^n$, we denote by $\mathcal{E}^+(A)$ the class of all $\mu \in \mathcal{E}^+$ *concentrated on* A, which means that $A^c := \mathbb{R}^n \setminus A$ is μ-negligible. (For closed A, $\mathcal{E}^+(A)$ consists of all $\mu \in \mathcal{E}^+$ with support $S(\mu) \subset A$.) Also fix an *external field* $f := -U^\vartheta$, where $\vartheta \in \mathfrak{M}^+$ is given. The problem in question is that on minimizing the *Gauss functional* $I_f(\mu)$, which sometimes is also referred to as the *f-weighted energy*, where

$$I_f(\mu) := \|\mu\|^2 + 2 \int f\, d\mu = \|\mu\|^2 - 2I(\mu, \vartheta)$$

and μ ranges over $\mathcal{E}^1(A) := \bigl\{\mu \in \mathcal{E}^+(A) : \ \mu(\mathbb{R}^n) = 1\bigr\}$. That is, *does there exist* $\lambda_{A,f} \in \mathcal{E}^1(A)$ *with*

$$I_f(\lambda_{A,f}) = \inf_{\mu \in \mathcal{E}^1(A)} I_f(\mu)? \tag{1}$$

The investigation of this problem, initiated by Gauss, is still of interest due to its important applications in various areas of mathematics (see e.g. Saff and Totik [5] and numerous references therein).

If $A := K$ is compact while $f|_K$ is finitely continuous, then $\lambda_{K,f}$ does exist, for $I_f(\cdot)$ is vaguely lower semicontinuous, whereas $\mathcal{E}^1(K)$ is vaguely compact [1, Section III.1, Corollary 3 to Proposition 15]. However, these arguments, based on the vague topology only, fail down if A is noncompact, and the problem becomes "rather difficult" (Ohtsuka [4, p. 219]). To examine problem (1) for noncompact A, we developed an approach based on the perfectness of κ_α, whence on *both the strong and vague topologies* on $\mathcal{E}$ (see [10, 11]). To this end, we need to impose on A and ϑ the following three requirements:

- *The cone $\mathcal{E}^+(A)$ is strongly closed, whence strongly complete.* (As shown in [10, Theorem 3.9], this in particular holds if A is closed or even *quasiclosed*. By Fuglede [2], the latter means that A can be approximated in outer capacity by closed sets. For the concepts of *outer* and *inner capacities*, see e.g. Landkof [3, Section II.2.6]. It is worth noting here that a quasiclosed set is *not* necessarily Borel.)
- $c_*(A) > 0$, *where $c_*(\cdot)$ stands for the inner capacity of a set; or equivalently $\mathcal{E}^1(A) \neq \varnothing$.*

- *$\vartheta\in\mathfrak{M}^+$ is bounded, i.e. $\vartheta(\mathbb{R}^n)<\infty$, and moreover*

$$\inf_{(x,y)\in S(\vartheta)\times A}|x-y|>0.$$

Then, the *inner κ_α-balayage* ϑ^A of ϑ to A can be defined as the unique bounded measure in $\mathcal{E}^+(A)$ such that $U^{\vartheta^A}=U^\vartheta$ n.e. on A, i.e. on all of A except for a set with $c_*(\cdot)=0$; see [10, Theorem 4.7(iii$_1$)]. (For the general theory of inner κ_α-balayage, we refer to [6, 7], cf. also [8, 9].) This implies that $I_f(\cdot)$ is strongly continuous on $\mathcal{E}^+(A)$, which is crucial to the analysis of problem (1), performed in [10, 11].

Theorem 1 (see [11, Theorem 2.6])**.** *For $\lambda_{A,f}$ to exist, it is necessary and sufficient that*

$$c_*(A)<\infty \quad \text{or} \quad \vartheta^A(\mathbb{R}^n)\geqslant 1. \tag{2}$$

By [7, Definition 2.1], $Q\subset\mathbb{R}^n$ is said to be *not inner α-thin at infinity* if

$$\sum_{j\in\mathbb{N}}\frac{c_*(Q_j)}{q^{j(n-\alpha)}}=\infty,$$

where $q\in(1,\infty)$ and $Q_j:=Q\cap\bigl\{y\in\mathbb{R}^n:\ q^j<|y|\leqslant q^{j+1}\bigr\}$. The inner κ_α-balayage of any $\mu\in\mathfrak{M}^+$ to such Q preserves its total mass [7, Corollary 5.3], whence the following corollary to Theorem 1 holds.

Corollary 2. *If A is not inner α-thin at infinity, then $\lambda_{A,f}$ exists if and only if $\vartheta(\mathbb{R}^n)\geqslant 1$.*

Theorem 3 (see [11, Theorem 2.10])**.** *Assume* (2) *is fulfilled, and moreover $\vartheta^A(\mathbb{R}^n)\leqslant 1$. Then*

$$\lambda_{A,f}=\begin{cases}\vartheta^A+c_{A,f}\gamma_A & \text{if } c_*(A)<\infty,\\ \vartheta^A & \text{otherwise},\end{cases} \tag{3}$$

where $c_{A,f}\in[0,\infty)$, while γ_A is the inner κ_α-equilibrium measure on A, normalized by $\gamma_A(\mathbb{R}^n)=c_*(A)$.

For the inner κ_α-equilibrium measure on the set A in question, see [9, Theorem 7.2] with $\kappa:=\kappa_\alpha$.

In the following Theorems 4 and 5, A is assumed to be *closed*. The *reduced kernel* $\breve{A}$ of A is the set of all $x\in A$ such that $c_*(A\cap U_x)>0$ for any neighborhood U_x of x in $\mathbb{R}^n$, cf. [3, p. 164].

Theorem 4 (see [11, Theorem 2.11])**.** *Under the requirements of Theorem* 3*, assume moreover that A^c is connected unless $\alpha<2$. Then, by virtue of the representation* (3) *and* [6, Theorems 7.2, 8.5],

$$S(\lambda_{A,f})=\begin{cases}\breve{A} & \text{if } \alpha<2,\\ \partial_{\mathbb{R}^n}\breve{A} & \text{otherwise.}\end{cases}$$

Theorem 5 (see [10, Theorem 2.22])**.** *If A is not α-thin at infinity and $\delta(\mathbb{R}^n)>1$, then $S(\lambda_{A,f})$ is compactly supported in A.* (Compare with Theorem 4. Note that $\lambda_{A,f}$ does exist, see Corollary 2.)

Theorems 4, 5 give an answer to the question raised by Ohtsuka in [4, p. 284, Open question 2.1].

References

[1] Bourbaki, N.: Integration. Chapters 1–6. Springer (2004)
[2] Fuglede, B.: The quasi topology associated with a countably subadditive set function. Ann. Inst. Fourier Grenoble **21**, 123–169 (1971)
[3] Landkof, N.S.: Foundations of Modern Potential Theory. Springer (1972)
[4] Ohtsuka, M.: On potentials in locally compact spaces. J. Sci. Hiroshima Univ. Ser. A-I **25**, 135–352 (1961)
[5] Saff, E.B., Totik, V.: Logarithmic Potentials with External Fields. Springer (2024)
[6] Zorii, N.: A theory of inner Riesz balayage and its applications. Bull. Pol. Acad. Sci. Math. **68**, 41–67 (2020)
[7] Zorii, N.: Harmonic measure, equilibrium measure, and thinness at infinity in the theory of Riesz potentials. Potential Anal. **57**, 447–472 (2022)
[8] Zorii, N.: Balayage of measures on a locally compact space. Anal. Math. **48**, 249–277 (2022)

[9] Zorii, N.: On the theory of capacities on locally compact spaces and its interaction with the theory of balayage. Potential Anal. **59**, 1345–1379 (2023)

[10] Zorii, N.: Minimum Riesz energy problems with external fields. J. Math. Anal. Appl. **526**, 127235 (2023)

[11] Zorii N.: Inner Riesz balayage in minimum energy problems with external fields. Constr. Approx., to appear, arXiv:2306.12788v2