Surfaces with flat normal connection in 4-dimensional space forms

Naoya Ando (Kumamoto University, Japan)

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The main contents of this talk are found in

[4] N. Ando, The equations of Gauss, Codazzi and Ricci of surfaces in 4-dimensional space forms, preprint, arXiv:2505.13874.

[5] N. Ando and R. Hatanaka, Surfaces with flat normal connection in 4-dimensional space forms, preprint, arXiv:2501.15780.

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1. The connection of the two-fold exterior power of the pull-back bundle

(N, h): a 4-dimensional Riemannian space form,

M: a Riemann surface,

$F : M \longrightarrow N$: a conformal immersion,

(u, v): local isothermal coordinates of M.

The induced metric g of M by F is represented as $g = e^{2\lambda}(du^2 + dv^2)$.

$T_1 := dF\left(\dfrac{\partial}{\partial u}\right)$, $T_2 := dF\left(\dfrac{\partial}{\partial v}\right)$,

N_1, N_2: normal vector fields of F satisfying

$$h(N_1, N_1) = h(N_2, N_2) = e^{2\lambda}, \quad h(N_1, N_2) = 0.$$

Suppose that N is oriented and that (T_1, T_2, N_1, N_2) gives the orientation. We set $e_1 := \dfrac{1}{e^\lambda}T_1,\ \ e_2 := \dfrac{1}{e^\lambda}T_2,\ \ e_3 := \dfrac{1}{e^\lambda}N_1,\ \ e_4 := \dfrac{1}{e^\lambda}N_2$, and

$$\Theta_{\pm,1} := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4),$$
$$\Theta_{\pm,2} := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2),$$
$$\Theta_{\pm,3} := \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

The two-fold exterior power $\bigwedge^2 F^*TN$ of the pull-back bundle F^*TN on M by F is decomposed into two subbundles $\bigwedge^2_\pm F^*TN$, and $\Theta_{\pm,1}$, $\Theta_{\pm,2}$, $\Theta_{\pm,3}$ form local orthonormal frame fields of $\bigwedge^2_\pm F^*TN$ respectively.

∇: the Levi-Civita connection of (N, h),

$\hat{\nabla}$: the connection of $\bigwedge^2 F^*TN$ induced by ∇.

Then $\hat{\nabla}$ gives connections of $\bigwedge^2_{\pm} F^*TN$ and we obtain

$$\hat{\nabla}_{T_1}(\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) = (\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) \begin{bmatrix} 0 & -W_{\pm} & -Y_{\mp} \\ W_{\pm} & 0 & \pm\psi_{\pm} \\ Y_{\mp} & \mp\psi_{\pm} & 0 \end{bmatrix},$$

$$\hat{\nabla}_{T_2}(\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) = (\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) \begin{bmatrix} 0 & \mp Z_{\pm} & \pm X_{\mp} \\ \pm Z_{\pm} & 0 & \mp\phi_{\mp} \\ \mp X_{\mp} & \pm\phi_{\mp} & 0 \end{bmatrix},$$

where

- $W_\pm$, $X_\pm$, $Y_\pm$, $Z_\pm$ are functions given by

  $$W_\pm = \alpha_2 \pm \beta_1, \ \ X_\pm = \alpha_2 \pm \beta_3, \ \ Y_\pm = \beta_2 \pm \alpha_1, \ \ Z_\pm = \beta_2 \pm \alpha_3$$

  and

  $$\begin{aligned} &\sigma(T_1, T_1) = \alpha_1 N_1 + \beta_1 N_2, \quad \sigma(T_1, T_2) = \alpha_2 N_1 + \beta_2 N_2, \\ &\sigma(T_2, T_2) = \alpha_3 N_1 + \beta_3 N_2 \end{aligned}$$

  for the second fundamental form σ of F, and

- $\phi_\pm$, $\psi_\pm$ are functions given by $\phi_\pm = \lambda_u \mp \mu_2$, $\ \psi_\pm = \lambda_v \mp \mu_1$, and μ_1, μ_2 are functions given by

  $$\nabla^\perp_{T_1} N_1 = \lambda_u N_1 + \mu_1 N_2, \quad \nabla^\perp_{T_2} N_1 = \lambda_v N_1 + \mu_2 N_2$$

  for the normal connection $\nabla^\perp$ of F.

2. The equations of Gauss, Codazzi and Ricci

$\hat{R}$: the curvature tensor of $\hat{\nabla}$,

L_0: the constant sectional curvature of N.

Then we have

$$\hat{R}(T_1, T_2)(\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) = (\Theta_{\pm,1}\ \Theta_{\pm,2}\ \Theta_{\pm,3}) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \pm L_0 e^{2\lambda} \\ 0 & \mp L_0 e^{2\lambda} & 0 \end{bmatrix}.$$

We can express the left hand side by $W_\pm$, $X_\pm$, $Y_\pm$, $Z_\pm$ and so on, and we obtain

$$W_\mp X_\pm + Y_\pm Z_\mp = L_0 e^{2\lambda} + (\phi_\pm)_u + (\psi_\mp)_v \quad \text{(the equations of Gauss, Ricci)},$$

$$\left.\begin{array}{l} (Y_\pm)_v \mp (X_\pm)_u = \pm W_\mp \phi_\pm - Z_\mp \psi_\mp, \\ (W_\mp)_v \pm (Z_\mp)_u = \mp Y_\pm \phi_\pm - X_\pm \psi_\mp \end{array}\right. \quad \text{(the equations of Codazzi)}$$

([4, 5]).

K: the curvature of g: $K = -e^{-2\lambda}(\lambda_{uu} + \lambda_{vv})$.

We say that the normal connection $\nabla^{\perp}$ of F is *flat*
if the curvature tensor $R^{\perp}$ of $\nabla^{\perp}$ vanishes, which is equivalent to $(\mu_1)_v = (\mu_2)_u$.

The twistor lifts of F are sections of the twistor spaces associated with the pull-back bundle F^*TN and locally given by $\Theta_{\pm,1}$.

The twistor lifts $\Theta_{\pm,1}$ of F are said to be *nondegenerate* (resp. *degenerate*) if for each $\varepsilon \in \{+,-\}$, $\hat{\nabla}_{T_1}\Theta_{\varepsilon,1}$ and $\hat{\nabla}_{T_2}\Theta_{\varepsilon,1}$ are linearly independent (resp. dependent) at each point of M.

Then we observe that the following conditions are mutually equivalent ([4]):

(a) the immersion F satisfies both $K \equiv L_0$ and $R^{\perp} \equiv 0$;

(b) the twistor lifts $\Theta_{\pm,1}$ of F are degenerate;

(c) $\Delta_{\pm} := W_{\mp}X_{\pm} + Y_{\pm}Z_{\mp} = 0$.

Remark

The twistor lifts $\Theta_{\pm,1}$ of F are nondegenerate if and only if $\Delta_\pm \neq 0$.

If we suppose $\Delta_\pm \neq 0$, then the equations of Codazzi are rewritten into $(\phi_\pm, \psi_\mp) = (A_\pm, B_\mp)$, where

$$\begin{bmatrix} A_\pm \\ B_\mp \end{bmatrix} := \mp\frac{1}{\Delta_\pm}\begin{bmatrix} -X_\pm & Z_\mp \\ \pm Y_\pm & \pm W_\mp \end{bmatrix}\begin{bmatrix} (Y_\pm)_v \mp (X_\pm)_u \\ (W_\mp)_v \pm (Z_\mp)_u \end{bmatrix}.$$

Using $A_\pm$, $B_\mp$, we can obtain characterizations of surfaces such that the twistor lifts are nondegenerate ([4]).

3. Surfaces with degenerate twistor lifts

- If F has a parallel normal vector field, then the second fundamental form σ of F satisfies the *linearly dependent condition*, that is, F satisfies

$$\cos\theta(\alpha_1, \alpha_2, \alpha_3) + \sin\theta(\beta_1, \beta_2, \beta_3) = 0$$

  for a function θ.

- If σ satisfies the linearly dependent condition, then $\nabla^{\perp}$ is flat.

- Suppose $K \neq L_0$. Then F has a parallel normal vector field if and only if σ satisfies the linearly dependent condition ([5]).

- If we suppose $K \equiv L_0$, then the linearly dependent condition of σ does not necessarily mean the existence of parallel normal vector fields ([5]).

Suppose that there exist nowhere zero functions $k_\pm$ satisfying

$$(W_\mp, Z_\mp) = k_\pm(-Y_\pm, X_\pm). \tag{1}$$

Then $\Delta_\pm = 0$ hold, that is, F satisfies both $K \equiv L_0$ and $R^\perp \equiv 0$.

By (1) and the equations of Codazzi, there exist functions $f_\pm$ satisfying

$$X_\pm = \pm\frac{(f_\pm)_v}{\sqrt{1+k_\pm^2}}, \quad Y_\pm = \frac{(f_\pm)_u}{\sqrt{1+k_\pm^2}}. \tag{2}$$

By the equation of Ricci, we obtain

$$(f_+)_u^2 + (f_+)_v^2 = (f_-)_u^2 + (f_-)_v^2 \ (=: B).$$

Therefore, if $B \neq 0$, then there exists a function ψ satisfying

$$\begin{bmatrix} (f_-)_u \\ (f_-)_v \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} (f_+)_v \\ (f_+)_u \end{bmatrix}. \tag{3}$$

Suppose that $X_\pm$, $Y_\pm$ satisfy $X_+^2 Y_-^2 - X_-^2 Y_+^2 \neq 0$.
Then σ does not satisfy the linearly dependent condition ([5]).

By the definitions of $W_\pm$, $X_\pm$, $Y_\pm$, $Z_\pm$, we have

$$W_+ + W_- = X_+ + X_-, \quad Y_+ + Y_- = Z_+ + Z_-.$$

Applying (1) to these relations, we obtain

$$k_+ = \frac{X_-^2 + Y_-^2 + X_+X_- + Y_+Y_-}{X_+Y_- - X_-Y_+},$$
$$k_- = -\frac{X_+^2 + Y_+^2 + X_+X_- + Y_+Y_-}{X_+Y_- - X_-Y_+}.$$

Applying (2) to these relations, we obtain $k_+ = \dfrac{Ck_- - A}{Ak_- + C}$, where

$$A := (f_+)_v(f_-)_u + (f_-)_v(f_+)_u \ (\neq 0), \quad C := (f_+)_u(f_-)_u - (f_+)_v(f_-)_v.$$

We have $A = B\cos\psi$, $C = -B\sin\psi$.

Applying these relations and $k_+ = \dfrac{Ck_- - A}{Ak_- + C}$ to the equations of Codazzi, we obtain

$$\begin{bmatrix} \gamma_u \\ \gamma_v \end{bmatrix} = -\begin{bmatrix} (\theta_-)_u \\ (\theta_-)_v \end{bmatrix} + \frac{\dfrac{\partial(f_+, \psi)}{\partial(u, v)}}{\dfrac{\partial(f_+, f_-)}{\partial(u, v)}} \begin{bmatrix} (f_-)_u \\ (f_-)_v \end{bmatrix}$$

$$-\frac{1}{A'} \begin{bmatrix} 2(f_+)_u(f_-)_u & A \\ A & 2(f_+)_v(f_-)_v \end{bmatrix} \begin{bmatrix} \lambda_u \\ \lambda_v \end{bmatrix}$$

([5]), where

- γ is a function satisfying $\gamma_u = \mu_1$, $\gamma_v = \mu_2$,
- θ_- is a function given by $\tan\theta_- = k_-$,
- $A' := (f_+)_v(f_-)_u - (f_-)_v(f_+)_u = -\dfrac{\partial(f_+, f_-)}{\partial(u, v)}$ $(\neq 0)$.

In addition, if $L_0 = 0$, then we obtain

$$\begin{bmatrix} \gamma_u \\ \gamma_v \end{bmatrix} = -\begin{bmatrix} (\theta_-)_u \\ (\theta_-)_v \end{bmatrix} + \frac{\dfrac{\partial(f_+, \psi)}{\partial(u, v)}}{\dfrac{\partial(f_+, f_-)}{\partial(u, v)}} \begin{bmatrix} (f_-)_u \\ (f_-)_v \end{bmatrix}.$$

Therefore there exists a function $\tilde{\xi}$ of one variable satisfying

$$\frac{\partial(f_+, \psi)}{\partial(u, v)} = \tilde{\xi}(f_-) \frac{\partial(f_+, f_-)}{\partial(u, v)}.$$

By this relation and (3), we obtain

$$\begin{bmatrix} \psi_u \\ \psi_v \end{bmatrix} = (\cos^2 \psi)\boldsymbol{a} + (\cos\psi \sin\psi)\boldsymbol{b} + (\sin^2 \psi)\boldsymbol{c}$$

([5]), where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are $\boldsymbol{R}^2$-valued functions constructed by f_- and $\tilde{\xi}(f_-)$.

4. Space-like or time-like surfaces in neutral or Lorentzian 4-dimensional space forms

In the case where N is neutral, analogous discussions and results are valid for space-like or time-like surfaces.

In the case where N is Lorentzian, analogous discussions and results are also valid for space-like or time-like surfaces.
However, in this case, we need the decomposition of the complexification of the two-fold exterior power of the pull-back bundle on a space-like or time-like surface.

(N, h): a 4-dimensional Lorentzian space form,

M: a Riemann surface,

$F : M \longrightarrow N$: a space-like and conformal immersion,

(u, v): local isothermal coordinates of M.

The induced metric g of M by F is represented as $g = e^{2\lambda}(du^2 + dv^2)$.

$T_1 := dF\left(\dfrac{\partial}{\partial u}\right)$, $T_2 := dF\left(\dfrac{\partial}{\partial v}\right)$,

N_1, N_2: normal vector fields of F satisfying

$$h(N_1, N_1) = -h(N_2, N_2) = e^{2\lambda}, \quad h(N_1, N_2) = 0.$$

Suppose that N is oriented and that (T_1, T_2, N_1, N_2) gives the orientation.

We set $e_1 := \frac{1}{e^\lambda}T_1,\ \ e_2 := \frac{1}{e^\lambda}T_2,\ \ e_3 := \frac{1}{e^\lambda}N_1,\ \ e_4 := \frac{1}{e^\lambda}N_2$, and

$$\Theta_1 := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + \sqrt{-1}e_3 \wedge e_4),$$

$$\Theta_2 := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + \sqrt{-1}e_4 \wedge e_2),$$

$$\Theta_3 := \frac{1}{\sqrt{2}}(\sqrt{-1}e_1 \wedge e_4 + e_2 \wedge e_3).$$

The complexification $\bigwedge^2 F^*TN \otimes \boldsymbol{C}$ of the two-fold exterior power of the pull-back bundle F^*TN on M by F is decomposed into two subbundles $\bigwedge^2_{\pm} F^*TN$ of complex rank 3, and

Θ_1, Θ_2, Θ_3 form a local frame field of $\bigwedge^2_{+} F^*TN$.

∇: the Levi-Civita connection of (N, h).
Then ∇ induces a connection $\hat{\nabla}$ of $\bigwedge^2 F^*TN \otimes \boldsymbol{C}$ naturally.
In addition, $\hat{\nabla}$ gives the connections of $\bigwedge^2_{\pm} F^*TN$ ([4]).
Then we obtain

$$\hat{\nabla}_{T_1}(\Theta_1\ \Theta_2\ \Theta_3) = (\Theta_1\ \Theta_2\ \Theta_3)\begin{bmatrix} 0 & -W & \sqrt{-1}Y \\ W & 0 & \psi \\ -\sqrt{-1}Y & -\psi & 0 \end{bmatrix},$$

$$\hat{\nabla}_{T_2}(\Theta_1\ \Theta_2\ \Theta_3) = (\Theta_1\ \Theta_2\ \Theta_3)\begin{bmatrix} 0 & \sqrt{-1}Z & X \\ -\sqrt{-1}Z & 0 & -\phi \\ -X & \phi & 0 \end{bmatrix},$$

where

$$W := \alpha_2 - \sqrt{-1}\beta_1, \quad X := \alpha_2 + \sqrt{-1}\beta_3,$$
$$Y := \beta_2 - \sqrt{-1}\alpha_1, \quad Z := \beta_2 + \sqrt{-1}\alpha_3$$

and

$$\phi := \lambda_u - \sqrt{-1}\mu_2, \quad \psi := \lambda_v + \sqrt{-1}\mu_1.$$

By these relations, we obtain

$$WX - YZ = L_0 e^{2\lambda} + \phi_u + \psi_v \ \text{(the equations of Gauss, Ricci)},$$
$$\left.\begin{aligned} Y_v + \sqrt{-1}X_u &= -\sqrt{-1}W\phi - Z\psi, \\ W_v + \sqrt{-1}Z_u &= -\sqrt{-1}Y\phi - X\psi \end{aligned}\right. \quad \text{(the equations of Codazzi)}$$

([4, 5]).

Referring to Sections 2, 3, we can obtain analogous results for space-like surfaces in N.

Remark

For a local complex coordinate $w = u + \sqrt{-1}v$, we have

$$\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1 = \frac{1}{2}((W + Z)\Theta_2 - \sqrt{-1}(X + Y)\Theta_3).$$

Whether $\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1$ vanishes or not is determined by F, and $\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1 = 0$ is equivalent to

$$W + Z = 0, \quad X + Y = 0.$$

In particular, if $\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1 = 0$, then $WX - YZ = 0$.
Therefore $\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1 = 0$ gives a special class of space-like surfaces such that the complex twistor lifts are degenerate, and
we can obtain a characterization of surfaces with $\hat{\nabla}_{\partial/\partial\overline{w}}\Theta_1 = 0$ ([4]).

M: a Lorentz surface,

$F : M \longrightarrow N$: a time-like and conformal immersion,

(u, v): local coordinates of M compatible with the paracomplex structure of M.

The induced metric g of M by F is represented as $g = e^{2\lambda}(du^2 - dv^2)$.

$T_1 := dF\left(\frac{\partial}{\partial u}\right)$, $T_2 := dF\left(\frac{\partial}{\partial v}\right)$,

N_1, N_2: normal vector fields of F satisfying

$$h(N_1, N_1) = h(N_2, N_2) = e^{2\lambda}, \quad h(N_1, N_2) = 0.$$

Suppose that N is oriented and that (T_1, T_2, N_1, N_2) gives the orientation.

We set $e_1 := \frac{1}{e^\lambda} N_1, \ \ e_2 := \frac{1}{e^\lambda} N_2, \ \ e_3 := \frac{1}{e^\lambda} T_1, \ \ e_4 := \frac{1}{e^\lambda} T_2.$

Then

$$\begin{aligned}
\overline{\Theta_1} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - \sqrt{-1} e_3 \wedge e_4), \\
\overline{\Theta_2} &:= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - \sqrt{-1} e_4 \wedge e_2), \\
\overline{\Theta_3} &:= \frac{1}{\sqrt{2}}(-\sqrt{-1} e_1 \wedge e_4 + e_2 \wedge e_3).
\end{aligned}$$

form a local frame field of $\bigwedge^2_- F^* TN$.

We obtain

$$\hat{\nabla}_{T_1}(\overline{\Theta_1}\ \overline{\Theta_2}\ \overline{\Theta_3}) = (\overline{\Theta_1}\ \overline{\Theta_2}\ \overline{\Theta_3}) \begin{bmatrix} 0 & -\sqrt{-1}W & -\sqrt{-1}Y \\ \sqrt{-1}W & 0 & -\sqrt{-1}\psi \\ \sqrt{-1}Y & \sqrt{-1}\psi & 0 \end{bmatrix},$$

$$\hat{\nabla}_{T_2}(\overline{\Theta_1}\ \overline{\Theta_2}\ \overline{\Theta_3}) = (\overline{\Theta_1}\ \overline{\Theta_2}\ \overline{\Theta_3}) \begin{bmatrix} 0 & Z & -X \\ -Z & 0 & -\sqrt{-1}\phi \\ X & \sqrt{-1}\phi & 0 \end{bmatrix},$$

where

$$W := \alpha_2 + \sqrt{-1}\beta_1, \quad X := \alpha_2 + \sqrt{-1}\beta_3,$$
$$Y := \beta_2 - \sqrt{-1}\alpha_1, \quad Z := \beta_2 - \sqrt{-1}\alpha_3$$

and

$$\phi := \lambda_u - \sqrt{-1}\mu_2, \quad \psi := \lambda_v - \sqrt{-1}\mu_1.$$

By these relations, we obtain

$$WX + YZ + L_0 e^{2\lambda} + \phi_u - \psi_v = 0 \quad \text{(the equations of Gauss, Ricci)},$$

$$\left.\begin{aligned} Y_v + \sqrt{-1}X_u &= -\sqrt{-1}W\phi - Z\psi, \\ W_v - \sqrt{-1}Z_u &= \sqrt{-1}Y\phi - X\psi \end{aligned}\right. \quad \text{(the equations of Codazzi)}$$

([4, 5]).

Referring to Sections 2, 3, we can obtain analogous results for time-like surfaces in N.

THANK YOU VERY MUCH

FOR YOUR ATTENTION!