From Maxwell's equations to relativistic Schrödinger equation via Schwartz Linear Algebra and Killing vector fields on the 2-sphere AGMA 2025

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Introduction

In this work, we develop a comprehensive mathematical framework unifying scalar Relativistic Quantum Mechanics with classical electromagnetic field theory by means of Schwartz-linear algebra. Building upon the foundations introduced by David Carfì in [1, 2, 3, 4], we construct a partial embedding of tempered scalar distributions into spaces of tempered vector-valued fields that carry natural Maxwellian structure.

The embedding operator $J_{(\eta,f)}$

The key object of our study is an embedding operator $J_{(\eta,f)}$, Schwartz-linear (therefore linear and continuous), that maps a large class of complex wave distributions

$$\psi \in V = \mathcal{S}'(\mathbb{M}^4, \mathbb{C})$$

into transverse vector fields

$$F_\psi \in W = \mathcal{S}'(\mathbb{M}^4, \mathbb{C}^3)$$

via spectral synthesis with polarization.

Specifically, the embedding is constructed using a transverse, right-handed, orthonormal polarization frame

$$f : k \mapsto f(k) = (r(k), s(k)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

defined on the dual space $\mathbb{M}_4^*$ minus Π_f, where Π_f is a singular plane and r is a normalized Killing vector field on the 2-sphere extended omogeneously to the whole dual of Minkowski space-time minus Π_f.

The de Broglie-Maxwell-Killing's basis

The de Broglie-Maxwell-Killing's basis

$$w : \mathbb{M}_4^* \setminus \Pi_f \to W : k \mapsto w_k := \eta_k(r(k) + is(k)),$$

where

$$\eta_k = e^{i\langle k,x\rangle}$$

defines the de Broglie basis in the space of complex tempered wave distributions, forms a Schwartz linearly independent system of circularly polarized plane waves, generating a vast subspace S of Schwartz-Maxwell electromagnetic field space W.

The map

$$J_{(\eta,f)} : \psi \mapsto J_{(\eta,f)}(\psi) = \int_{\mathbb{M}_4^*} (\psi)_\eta w$$

embeds scalar wave distributions into the Maxwell-Schwartz field space, provided that the complex wave distribution ψ admits a momentum representation $(\psi)_\eta$ vanishing around the singular plane Π_f.

Eigenstructures of quantum observables

We show that the above embedding preserves eigenstructures of quantum observables diagonal on η.
The momentum operators

$$\hat{p} = -i\hbar\nabla$$

and energy operators

$$\hat{E} = i\hbar\partial_0,$$

of both spaces V and W, act compatibly through $J_{(\eta,f)}$, and w_k are simultaneous eigenfunctions of $\hat{p}$ and curl in W, with eigenvalues $\hbar\vec{k}$ and $|\vec{k}|$, respectively.

The operator

$$\hbar\,\mathsf{curl}$$

is therefore identified with the momentum magnitude operator on the subspace

$$S = {}^{\mathcal{S}}\mathrm{span}(w)$$

of the Maxwell-Schwartz space W.

General embeddings $J_{(\beta,f)}$

Furthermore, the theory is extended to general embeddings $J_{(\beta,f)}$, constructed from arbitrary Schwartz bases β and smooth frame fields

$$f : D \to \mathbb{C}^3$$

with the same domain of β. These embeddings commute with all observables diagonal in the basis β, yielding a functorial structure.

We also define position-space embeddings

$$I_j \, (j = 1, 2, 3)$$

using Dirac delta basis and demonstrate that they preserve position eigenstates. This duality between frequency-space and position-space embeddings reveals a deep symmetry between quantum representations.

As an example, a geometric interpretation is introduced via the use of Frenet frames along spatial curves, allowing for the representation of localized electromagnetic waves carrying geometric signatures of trajectories. Fields such as

$$\delta_0 \circ \gamma \cdot f \circ \gamma : t \mapsto \delta_{\gamma(t)} \cdot f(\gamma(t))$$

are shown to encode the curve γ via the support and polarization f.

The Maxwell-Schrödinger equation in W

The relativistic Schrödinger equation for photons, in tempered distribution space, is recovered in the form

$$\hat{E}\psi = c|\hat{p}|\psi.$$

We show that in our subspace S the curl Maxwell's equations can be synthesized into the same Schrödinger's form equation

$$\hat{E}F = c|\hat{p}|F,$$

where $\hat{E}$ is the energy operator in our space W, perfectly analogous to the energy operator in the space of complex tempered distribution, $\hat{p}$ is the momentum operator in W whose magnitude operator equals the operator $\hbar$curl on the subspace S.

This shows that any wave distribution ψ, with a momentum representation vanishing around the singular plane Π_f, can be smoothly interpreted as encoding an electromagnetic-type field.

A wave distribution ψ solves the massless relativistic Schrödinger equation if an only if the corresponding electromagnetic-type field F_ψ solves the massless Schrödinger-Maxwell equation in W.

Analogously, we construct a faithfull representation of the relativistic Schrödinger equation for massive particles in our space W, showing that each wave distribution state of a massive particle (complex field) can be smoothly interpreted as an electromagnetic-like field in W.

Delta distributions

$$\psi = \delta_0(x \mp ct)$$

are proven to be solutions of photons equation with spectral support positive or negative, corresponding to right-moving and left-moving massless particles, respectively.

The relation

$$m = \hbar|\vec{k}|/c$$

defines the relativistic mass of a photon as a function of spectral content. On the other hand, the dispersion relation of the massive plane wave fields satisfying the Maxwell-Schrödinger equation, is given by the Einstein's energy relation.

Relativistic Schrödinger Equation for Massive Particles

In the massless case (photons), the relativistic Schrödinger equation takes the form:

$$\hat{E}\psi = c|\hat{p}|\psi$$

which corresponds to the linear dispersion relation $E = c|\vec{p}|$.

To describe **massive spin-0 particles**, we adopt the Einstein dispersion relation:

$$E^2 = |\vec{p}|^2 c^2 + m_0^2 c^4 \quad \Rightarrow \quad E = \sqrt{|\vec{p}|^2 c^2 + m_0^2 c^4}.$$

Operator Formulation

We define the relativistic Hamiltonian operator as:

$$\hat{H}_{m_0} := \sqrt{c^2 \hat{p}^2 + (m_0^2 c^4)\mathbb{I}_V}$$

which acts on the de Broglie basis by:

$$\hat{H}_{m_0}(\eta_k) = H_{m_0}(\hbar \vec{k}) \cdot \eta_k, \quad \text{with} \quad H_{m_0}(\vec{p}) = \sqrt{|\vec{p}|^2 c^2 + m_0^2 c^4}$$

The above relativistic Hamiltonian operator is **Schwartz-linear**, because it is constructed by multiplying a smooth real function times the Schwartz basis η in momentum space.

Note on Linearity of Square Root Operators

The square root here is not a multiplicative function but rather an operator-valued spectral function. As an example, consider the linear endomorphism $x \mapsto mx$ on $\mathbb{R}$. Its operator square root is $x \mapsto \sqrt{m}x$, which is still linear. What would be nonlinear is the function $x \mapsto \sqrt{mx}$, which is not our context. Hence, the square-root operator applied to

$$c^2(\hbar\hat{K})^2 + (m_0^2c^4)\mathbb{I}_V,$$

and in the next analysis applied to

$$c^2(\hbar\,\mathrm{curl})^2 + (m_0^2c^4)\mathbb{I}_W,$$

remains linear.

Embedding into the Maxwell Space

Let $F_\psi \in W$ be the image of a scalar wave ψ under the vectorial embedding:

$$J_{(\eta,f)}(\psi) := \int_{\mathbb{M}_4^*} (\psi)_\eta \; \eta \cdot (r + is) =: F_\psi$$

We define the Hamiltonian operator on W as:

$$\hat{\mathcal{H}}_{m_0} := \sqrt{c^2(\hbar\,\mathrm{curl})^2 + (m_0^2 c^4)\,\mathbb{I}_W}$$

Then, F_ψ satisfies the Maxwell-Schrödinger equation in W,

$$i\hbar\partial_t F = \hat{\mathcal{H}}_{m_0} F$$

if and only if ψ satisfies the relativistic Schrödinger equation in V.

Spectral Behavior

Since

$$\vec{\nabla} \times w_k = |\vec{k}| w_k,$$

we find:

$$\hat{\mathcal{H}}_{m_0}(w_k) = H_{m_0}(\hbar\vec{k}) \cdot w_k,$$

for every $k \in \mathbb{M}_4^* \setminus \Pi_f$.

Conclusion

Recapitulating, the relativistic dynamics of massive spin-0 particles are captured, in the Maxwell field subspace $S \subset W$, by the equation:

$$\boxed{\hat{E}(F) = \sqrt{c^2(\hbar\,\mathrm{curl})^2 + (m_0^2 c^4)\mathbb{I}_W}\ (F).}$$

The massive Hamiltonian operator is Schwartz-linear and spectrally defined, providing a consistent extension of our theory from massless to massive quantum states.

Schwartz Framework as a Platform for Unified Dynamics

A Unified Architecture

We propose that the Maxwell–Schwartz–Minkowski structure

$$W := \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$$

serves as a central object for the unification of Einstein's Relativity, electromagnetism, and relativistic quantum mechanics on Minkowski space-time.

- **Relativistic Geometry:** Encoded in the manifold $\mathbb{M}_4$ and its tensor structures, including the Minkowski metric $\boldsymbol{\eta}$, which determines differential operators and Killing fields.
- **Complex Quantum Dynamics:** Encoded in the space of tempered scalar distributions $\mathcal{S}'(\mathbb{M}_4, \mathbb{C})$, with quantum observables and spectral structures realized via Schwartz-linear Algebra.
- **Vectorial Embedding and Quantum Fields:** The injections

$$J : \mathcal{S}'(\mathbb{M}_4, \mathbb{C}) \longrightarrow W$$

map scalar states into electromagnetic-like vector fields, compatible with the operator structure of Maxwell's and Schrödinger-type equations.

Extension to Curved Space-Time

The structure W remains viable under a metric deformation

$$\boldsymbol{\eta} \mapsto g,$$

upon space-time, enabling the inclusion of gravitational fields. Differential operators such as curl_g and Laplace–Beltrami derivatives deform consistently within W, providing a gravitationally-aware Maxwell-Schrödinger Hamiltonian operator:

$$\hat{H}_g := \sqrt{c^2(\hbar\,\mathrm{curl}_g)^2 + (m_0^2 c^4)\mathbb{I}_W}.$$

Conclusion

This layered architecture allows us to view:

- $(\mathbb{M}_4, g)$ as the relativistic pseudo-Riemannian geometric background;
- $\mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ as the scalar quantum platform;
- W as the host for Maxwell-Quantum Field Theory;
- metric tensor g as the vehicle for including General Relativity.

Thus, Schwartz-linear algebra not only provides a language for distribution waves and observables, but also a vessel for unifying field-theoretic geometry and relativistic quantum theory.

This work lays a foundation for a full spectral theory of relativistic fields within tempered distribution spaces, connecting canonical Quantum Mechanics, Maxwell's equations, and geometric field structures under a unified, mathematically rigorous umbrella.

References

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